**Bounds on local averages of one-dimensional electrical conductivity distributions**

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**INTRODUCTION**

All methods of geophysical inversion, which try to interpret real data, are methods of optimization. Whereas traditional approaches search for the model that minimizes the misfit between measured and modelled data, more recent approaches pre-assign the misfit and place emphasis on the construction of the model that extremizes a model property of geophysical interest. Prominent representatives of the latter strategy are the minimum-structure models (e.g. Constable, Parker & Constable 1987; Smith & Booker 1988) and the models leading to maximum-depth rules (e.g. Smith 1959, 1960; Parker 1974, 1975). The actual structure of the extremal models is in general of subordinate interest. What is learnt from the inversion is the extremal value of the model property under investigation, because this number is a bound, which all other competing—and possibly more realistic—models have to satisfy.

This paper considers the simplest problem of magnetotellurics, where the electrical conductivity \( \sigma \) depends on depth \( z \) only. For any finite set of data, even if it is accurate, point estimates of the electrical conductivity become meaningless, since at a specified depth level one may introduce either a thin highly conducting sheet or a thin insulating layer without changing the fit to the data. Therefore at a specified depth the conductivity may range between zero and infinity. Of geophysical interest, however, are estimates of the conductivity in a given depth range rather than point estimates. If the assigned depth range is sufficiently extended and shallow, the longer periods may be inconsistent with a depth range that is completely filled with material of the highest conductivity, or—on the other hand—the observed damping may be inconsistent with an extended poorly conducting layer. Therefore the average conductivity in the depth range considered can be constrained by the data. The possibility of putting constraints
on linear averages of the conductivity is in accord with the fact that the inverse problem for the conductance (i.e. integrated conductivity) is well-posed (Berdichevskiy & Dmitriev 1992, pp. 198-201).

For a given set of $M$ frequency-dependent surface impedances, we construct those extremal models that maximize or minimize the arithmetic average of $\sigma$ in the given depth range $z_1 \leq z \leq z_2$. Moreover, $\sigma$ may be subjected to the a priori constraints $\sigma_- \leq \sigma(z) \leq \sigma_+$, with $\sigma_-$ and $\sigma_+$, prescribed.

Problems of this kind have been treated previously by Oldenburg (1983) and Dosso & Oldenburg (1989). After discretizing the conductivity structure, the authors reduce the problem of determining the bounds to a problem in non-linear programming. Starting with an initial guess and linearizing the functional, which maps the conductivity onto the data, the problem is solved iteratively by a sequence of linear programming problems.

The special problem of extremizing the conductivity integrated over the surface $z_1=0$ and the level $z_2$ for the unconstrained case $\sigma_-=0$, $\sigma_+ = \infty$ has been considered by Weidelt (1985; hereafter W1). W1 explores, in a fully non-linear treatment, the exact structure of the extremal models for a small number of data. The present paper extends the results of W1 by assuming a depth range $z_1 \leq z \leq z_2$ and finite $a$ priori bounds $\sigma_+ \geq 0$ and $\sigma_- \leq \infty$. A simple structure is obtained only in the one-frequency case, whereas partial results for the general frequency case are presented in Section 4. Applications to synthetic and real data are given both in Sections 3 and 4. The Appendix lists useful results on the electromagnetic inversion.

The methods of Oldenburg (1983) and Dosso & Oldenburg (1989) on the one hand and those of W1 on the other are complementary: the former approximate and linear treatment, the exact structure of the extremal models degenerate in the unconstrained case) and provides a priori bounds to a problem in non-linear programming. Problems of this kind have been treated previously by Oldenburg (1983) and Problems of this kind have been treated previously by Oldenburg (1983) and Dosso & Oldenburg (1989). After discretizing the conductivity structure, the authors reduce the problem of determining the bounds to a problem in non-linear programming. Starting with an initial guess and linearizing the functional, which maps the conductivity onto the data, the problem is solved iteratively by a sequence of linear programming problems.

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2 BASIC EQUATIONS AND NECESSARY EXTREML CONDITIONS

Attention is confined to a 1-D conductivity profile $\sigma(z)$, positive downwards, and a uniform inducing magnetic field in the y-direction. Assuming a time factor $e^{i\omega t}$, $\omega > 0$, throughout, the field equations in the quasi-static limit are

$$E_x(z,\omega) = -io\mu_0 H_y(z,\omega), \quad H_y(z,\omega) = -\sigma(z)E_x(z,\omega),$$

where the prime denotes differentiation with respect to $z$. They lead to the differential equation

$$f''(z,\omega) = io\mu_0 \sigma(z)f(z,\omega),$$

with

$$f(z,\omega) = -E_x(z,\omega)/E_x(0^-,\omega),$$

where the discontinuity of $E_x$ due to a possible thin conducting surface sheet has been taken into account. The boundary conditions imposed on $f(z)$ are

$$f'(0^-) = -1, \quad f'(\infty) = 0.$$ (2)

In the following, we use Schmucker's response function (Schmucker 1970, p. 69):

$$c(\omega) = \frac{E_x(0,\omega)}{io\mu_0 H_y(0^-,\omega)} = f(0,\omega),$$

with $c = g - ih = |c| \exp(-i\psi)$, $g, h > 0$, which is related to the apparent resistivity $\rho_a$ and impedance phase $\varphi$ by

$$\rho_a = \omega \mu_0 |c|^2, \quad Z = io\mu_0 c, \quad \varphi = 90^\circ - \psi.$$ (3)

A set of $M$ frequencies $\omega_j$, $j \in [1, M]$ is considered, with the measured responses $c_j = c(\omega_j)$ being either exact or corrupted by noise with the standard deviations $s_j$. In addition, let $c_j (\sigma) = f(0, \omega_j)$ be the data functional, i.e. the result of solving (1) with the boundary conditions (2) for the conductivity profile $\sigma(z)$ and the frequency $\omega_j$. Hence $\sigma(z)$ is an acceptable model if, in the case of exact data,

$$c_j = c_j (\sigma), \quad j = 1, \ldots, M,$$ (3)

or if, in the case of noisy data, the soft $\chi^2$-bound

$$\sum_{j=1}^M |c_j - c_j (\sigma)|^2 / A^2 \leq B$$ (4)

is satisfied, where $B = \chi^2_{M\sigma}$ is the threshold, which for $2M$ degrees of freedom is exceeded with probability $\alpha$.

Apart from degenerate data, for example

$$c_j = \frac{a}{b + i\omega_j}, \quad a > 0, b \geq 0,$$

for which in the case of exact data and $M > 1$ only a single conductivity model exists, there will be a whole family of acceptable conductivity profiles. After pre-assigning a depth range $z_1 \leq z \leq z_2$ and an $a$ priori conductivity range

$$\sigma_- \leq \sigma(z) \leq \sigma_+, \quad 0 \leq z < \infty,$$ (5)

we will try to find the model that minimizes or maximizes

$$\sigma(z_1, z_2) = \frac{1}{\Delta} \int_{z_1}^{z_2} \sigma(z) dz,$$

subject to the constraints (3) [or (4)] and (5). The extremal averages are $\sigma_{\min}(z_1, z_2)$ and $\sigma_{\max}(z_1, z_2)$. Hence the objective function to be minimized is

$$Q[\sigma] = \int_0^{\infty} w(z) \sigma(z) dz,$$

with the weight function

$$w(z) = \begin{cases} 0, & z \notin (z_1, z_2) \\ +1/\Delta, & z \in (z_1, z_2), \quad Q \rightarrow +\sigma_{\min}(z_1, z_2) \\ -1/\Delta, & z \in (z_1, z_2), \quad Q \rightarrow -\sigma_{\max}(z_1, z_2). \end{cases}$$ (6)

The constraints (3) to (5) are taken into account by Lagrangian multipliers; see for example Avriel (1976) for a concise treatment. In the case of exact data (3) the Lagrange function is

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$L[\sigma] = Q[\sigma] + \Re \sum_{j=1}^{M} \lambda_j (c_j[\sigma] - c_j) + \int_{0}^{\infty} \left[ \mu_+(z)\{\sigma(z) - \sigma_+\} + \mu_-(z)\{\sigma_- - \sigma(z)\} \right] dz,$  

(7)

where $\Re$ denotes the real part. In the case of noisy data (4) the second term on the right-hand side is replaced by

$$
\Lambda \left\{ \sum_{j=1}^{M} |c_j[\sigma] - c_j|^2 |z|^j - B \right\}.
$$

(8)

Real and imaginary parts of the complex ordinary Lagrangian multipliers $\lambda_j$, which enforce equality constraints, are unrestricted in sign, whereas the generalized Lagrangian multipliers $\Lambda$ and $\mu_\pm(z)$, which account for inequality constraints, are sign-restricted and non-negative in the present definitions. In particular, these multipliers are zero whenever the constraints are inactive, and non-negative if the constraints are binding.

Therefore the expressions (8) and

$$
\mu_\pm(z)[\sigma(z) - \sigma_{\pm}], \quad 0 < z < \infty
$$

always vanish. The Lagrangian multipliers $\lambda_j$ and $\Lambda$ are closely related to the sensitivity of the minimum value $Q_0$ of $Q[\sigma]$ to changes in the data. Let $c_j = g_j - ih_j$. Then

$$
\Re \lambda_j = - \frac{\partial Q_0}{\partial g_j}, \quad \Im \lambda_j = - \frac{\partial Q_0}{\partial h_j}, \quad \Lambda = - \frac{\partial Q_0}{\partial B},
$$

(9)

where $\Im m$ denotes the imaginary part. The last equation expresses the obvious fact that an increase in the active $\chi^2$-bound leads to a further decrease in $Q_0$. The functions $\mu_\pm(z)$ describe the sensitivity of $Q_0$ to local changes of the conductivity bounds $\sigma_{\pm}$ at position $z$. If in a small range $\delta z$ around $z$ the bounds $\sigma_{\pm}$ are increased to $\sigma_{\pm} + \delta \sigma_{\pm}$, then with

$$
\partial \sigma_{\pm}(z) = \frac{\partial Q_0}{\partial \sigma_{\pm}(z)},
$$

(10)

i.e., if $\sigma(z) = \sigma_{\pm}$, implying $\mu_{\pm}(z) \geq 0$, an increase of $\sigma_{\pm}$ will not lead to a deeper minimum $Q_0$, whereas for $\sigma(z) > \sigma_{\pm}$, implying $\mu_{\pm}(z) < 0$, the change of an inactive bound does not affect $Q_0$. A similar interpretation holds for $\sigma(z)$ and $\sigma_{\pm}$.

The sensitivity of $Q_0$ to a global change of $\sigma_{\pm}$ is given by

$$
\int_{0}^{\infty} \mu_\pm(z) dz = \frac{\partial Q_0}{\partial \sigma_{\pm}}.
$$

(11)

As a necessary extremal condition, the first variation of the Lagrangian $L[\sigma]$ with respect to $\sigma(z)$ has to vanish. Whereas the first and third terms on the right-hand side of (7) are linear in $\sigma$ and pose no problems, the first variation of the non-linear data functional $c_j[\sigma]$ has to be expressed in terms of its Fréchet derivative $F_j(z)$ defined by

$$
\delta c_j[\sigma] = \int_{0}^{\infty} F_j(z) \delta\sigma(z) dz, \quad (11)
$$

with

$$
F_j(z) = -i \omega \mu_j f_j^*(z), \quad (12)
$$

where $f_j(z) = f(z, \omega_j)$ is the solution of (1) with the boundary conditions (2) (e.g. Parker 1977). Hence $\delta L[\sigma] = 0$ implies, for the exact data case (3),

$$
w(z) + \Re \sum_{j=1}^{M} \lambda_j F_j(z) + \mu_+(z) - \mu_-(z) = 0. \quad (13)
$$

In the case of noisy data, $\lambda_j$ in (13) is replaced by

$$
\bar{\lambda}_j = 2 \Lambda (c_j[\sigma] - c_j)^* |z|^j, \quad (14)
$$

where $^*$ denotes the complex conjugate.

In what follows, the necessary condition (13) is used to define the control function

$$
D(z) = w(z) + \Re \sum_{j=1}^{M} \lambda_j F_j(z) = \mu_-(z) - \mu_+(z). \quad (15)
$$

which states that

$$
D(z) \geq 0, \quad \text{when } \sigma(z) = \sigma_-, \quad D(z) = 0, \quad \text{when } \sigma_- < \sigma(z) < \sigma_+, \quad \text{D(z) \leq 0, \quad \text{when } \sigma(z) = \sigma_+.} \quad (16)
$$

It is stressed that—in exceptional situations—only the weak condition $D(z) = 0$, rather than strict positivity or negativity, can be achieved in a depth interval where a conductivity constraint is active; see, for instance, the simple examples given in Section 3.2.3.

The process of model construction therefore consists in selecting a model, within the class of models satisfying the data in the sense of (3) or (4), for which there exists a linear combination $D(z)$ of its Fréchet derivatives that, according to (16), is non-negative (non-positive) where $\sigma(z)$ attains its lower (upper) bound.

The problems with this prescription are at least three-fold.

(1) It does not lead to an immediate model construction, since in general $\sigma(z)$ has to be determined iteratively on the basis of the information on $\sigma(z)$ obtained from the sign changes of $D(z)$.

(2) There might be more than one model satisfying this necessary condition. In order to single out the extremal model, one has to be sure that all these admissible models are known.

(3) The structure of possible extremal models is not known at the outset: does it consist only of discrete layers or do continuous conductivity variations occur in addition? How many layers are required?

Despite these complications, satisfactory model constructions are possible in many cases, since it turns out that in most instances the conductivity only flips between the extremes $\sigma_-$ and $\sigma_+$. This holds in particular for the modest one-frequency case, which is considered in detail in the next section. Moreover, it is often easy to find the pertinent extremal model for small $z_1$ and $z_2$. By gradually deforming this solution and monitoring the change of $D(z)$, it is possible to decide for which parameter combination $(z_1, z_2)$ the type of the model has to change, for example where a conducting layer at the surface or at $z_1$ emerges or disappears, where two conducting layers coalesce, or where a continuous conductivity variation is required in some section of the model. These changes, of course, reflect the full non-linearity, which we take into account.
3 Extremal Averages for One Frequency

3.1 General Structure of the Extremal Models

If \( \sigma_- \) and \( \sigma_+ \) are suitably chosen such that feasible models exist (Section 3.2.2), the one-frequency extremal models have the property that \( \sigma(z) \) only attains the values \( \sigma_- \) and \( \sigma_+ \). Similarly to Wi, this is proved by contradiction by assuming that there exists a depth interval \( (a, b) \) completely inside or outside \( z_1, z_2 \) such that \( \sigma_- < \sigma(z) < \sigma_+ \) for \( z \in (a, b) \). Then, according to (16), \( D(z) \) and all its derivatives vanish identically for \( z \in (a, b) \). By dropping the subscript \( j \) identifying the frequency for the case \( M = 1 \), and using the fact that \( w(z) \) is piecewise constant, the first and second derivatives of \( D(z) \) yield, with reference to (15),

\[
\mathcal{R}_e[\lambda F(z)] = 0, \quad \mathcal{R}_e[\lambda F'(z)] = 0.
\]

These two homogeneous linear equations for \( \lambda \) have to satisfy the compatibility condition

\[
\mathcal{F}_m[F(z)/F'(z)] = 0.
\]

However, from (12) and (1) it follows that

\[
\mathcal{F}_m[F(z)/F'(z)] = -\mathcal{F}_m[\omega \mu_0 \sigma(z)c(z) + 1/c(z)]
\]

\[
= -\{\omega \mu_0 \sigma(z)g(z) + h(z)/(c(z))^2\} < 0,
\]

since \( g(z) \) and \( h(z) \) are positive as the real part and negative imaginary part of the response function \( c \) at level \( z > z_1 \):

\[
ct(z) = -f(z)f'(z) = g(z) - ih(z).
\]

Hence the compatibility condition cannot be satisfied and no continuous conductivity section exists. This conclusion holds for both exact and noisy data.

3.2 Unconstrained Conductivity Models

3.2.1 General Features

In Section 3.2 it is assumed that no constraint (except non-negativity) is imposed on \( \sigma(z) \), such that \( \sigma_- = 0, \sigma_+ = \infty \). According to Section 3.1 the extremal models therefore consist of a stack of \( K \) thin sheets of finite conductance \( \tau_k \) at level \( \zeta_k \),

\[
\sigma(z) = \sum_{k=1}^K \tau_k \delta(z - \zeta_k),
\]

terminated possibly be an additional perfectly conducting sheet at \( \zeta_{K+1} \). As yet, \( K \) is unspecified. A thin sheet is the limit of a conducting layer, for which the conductance increases to infinity and at the same time the thickness shrinks to zero, such that the conductance, as the product of conductivity and thickness, remains finite.

From (1), (17), and the definition (12), it follows that between adjacent sheets \( f(z) \) varies linearly and \( F(z) \) quadratically, and across sheets they show a jump in their derivatives. Taking \( \tau \) and \( \zeta \) as a generic pair of sheet parameters, the jump relations are

\[
f'(\zeta^+ - f'(\zeta^-) = \omega \mu_0 \sigma \tau \delta(z - \zeta).
\]

\[
F'(\zeta^+ - F'(\zeta^-) = 2\omega \mu_0 \sigma \tau F(z).
\]

From \( \partial \sigma / \partial \tau = \delta(z - \zeta) \) and \( \partial \sigma / \partial \zeta = -\tau \delta'(z - \zeta) \) it follows by using the definition of the Fréchet derivative (11) and the jump relation (18) that

\[
\frac{\partial \sigma(z)}{\partial \tau} = F(z) = -i \omega \mu_0 \sigma \tau F(z).
\]

\[
\frac{\partial \sigma(z)}{\partial \zeta} = \tau F'(z) = [f'(\zeta^-)]^2 + [f'(\zeta^+)]^2, \quad \tau < \infty,
\]

\[
\frac{\partial \sigma(z)}{\partial \zeta} = [f'(\zeta^-)]^2, \quad \tau = \infty.
\]

Eq. (21) was obtained after integration by parts using

\[
\int_{-\infty}^{\infty} F(z)\delta(z - \zeta) dz = \int_{-\infty}^{\infty} [F'(\zeta^-) + F'(\zeta^+)]/2
\]

\[
= F(\zeta).
\]

The depth levels that are distinguished are \( z_1, z_2 \) and \( z = 0 \), the latter as the plane of observation and natural boundary of the conductor. Sheets at these positions cannot move freely and are defined by their conductance only. On the other hand, a mobile sheet has its position as the second free parameter. An exception is the possibly existing final sheet of infinite conductance, which again is defined by only one parameter.

The following three properties of one-frequency extremal models are easily derived.

(1) No mobile sheet of finite conductance \( \tau \) can exist at \( \zeta \neq z_1, z_2 \). A final mobile sheet of infinite conductance, however, may occur.

(2) The extremal model for \( \sigma_{\text{max}} \) cannot have a mobile sheet at \( \zeta \in (z_1, z_2) \), whereas at most one mobile sheet appears in the models for \( \sigma_{\text{min}} \).

(3) The extremal models for \( \sigma_{\text{max}} \) terminate with a mobile sheet of infinite conductance at \( z = \zeta_{\text{in}} \) below a sheet at \( z = z_{\text{in}} \), whereas the extremal models for \( \sigma_{\text{min}} \) end with an insulator below a sheet at \( z = z_2 \).

Before presenting the proofs, the necessary extremal conditions outlined in Section 2 have to be briefly reviewed for the present unconstrained conductivities. Since only a lower bound (non-negativity) is imposed on \( \sigma(z) \), the Lagrangian multiplier function \( \mu(z) \) vanishes identically and the control function \( D(z) \), defined in (15), is non-negative. In particular it has to vanish at a position \( \zeta \) of a thin sheet, since the non-negativity constraint is not active there. If the sheet is mobile, the average slope of \( D(z) \) vanishes at \( \zeta \), thus avoiding negative values (double-zero). Therefore necessary conditions for a mobile sheet of finite \( \tau \) are

\[
D(z) - w(z) + \mathcal{R}_e[\lambda F(z)] = 0, \quad \overline{D(z)} = \mathcal{R}_e[\lambda \overline{F(z)}] = 0,
\]

expressing just the insensitivity of the minimum \( Q_0 \) to small changes of \( \tau \) and \( \zeta \). A perturbation \( \delta \tau \), for instance, affects \( Q_0 \) in two ways: it perturbs \( Q_0 \) by \( \delta Q_0^{(2)} = \delta \zeta \delta \tau \), and it perturbs \( c \) according to (20) for \( \delta c = F(\zeta) \delta \tau \). Since the response \( c \) has to remain unchanged, we have to calculate the modification \( \delta Q_0^{(2)} \) resulting from the response \( c - \delta c \). The first two equations of (9) immediately yield \( \delta Q_0^{(2)} = -\mathcal{R}_e[\lambda (\delta F)] = \mathcal{R}_e[\lambda F(z)] \delta \tau, \) such that according to (23) \( \delta Q_0^{(2)} + \delta Q_0^{(0)} = 0. \)
Proof of Proposition 1

One has simply to note that the existence of a solution depends on the compatibility condition \( \mathcal{J}_{\text{in}}[F(z)/F(z)] = 0 \). From (21), (20) and (18) it follows, however, that

\[
F(z)F(\eta) = [f'(\zeta^+) + f'(\zeta^-)]f(\zeta) = -(1/c^+ + 1/c^-),
\]

where \( c^+ \) and \( c^- \) are the responses below and above the sheet, which both have a negative imaginary part. Therefore the compatibility condition cannot be satisfied.

A final sheet with \( \zeta = \infty \) at \( \zeta \) can exist, since in view of the fact that \( F(\zeta) = 0 \) the first equation of (23) is satisfied automatically. For the second equation, (22) has to be used. If the sheet adjacent to \( \zeta \) is at \( \zeta = \eta < \zeta \), then

\[
f'(\zeta^-) = f'(\eta^-) = -f(\eta)(\zeta - \eta),
\]

such that with reference to (12) and (22) the second equation of (23) is cast into the convenient form

\[
\mathcal{J}_{\text{in}}[F'(\eta)] = 0,
\]

which we will use in the next section.

Proof of Proposition 2

We first show that \( \sigma_{\text{max}} \) has no mobile sheet at \( \zeta \in (z_1, z_2) \). Assuming a sheet at \( z = z_1 \) (cf. Proposition 3) the conditions to be satisfied are

\[
\Re{\lambda F'(\zeta)} = 1/\Delta,
\]

\[
\Re{\lambda F'(\zeta)} = 0,
\]

\[
\Re{\lambda F(z_2)} = 1/\Delta.
\]

These three linear equations for the complex multiplier \( \lambda \) require the compatibility condition

\[
\mathcal{J}_{\text{in}}[F'(\eta)](F(z_2) - F(\zeta)) = 0.
\]

From (A7) it follows that

\[
f(z_2) = f(\zeta)(1 - d/c^+),
\]

with \( d = z_2 - \zeta \). Therefore

\[
F(z_2) - F(\zeta) = \frac{1/c^+ + 1/c^-}{2d/c^+ - (d/c^+)^2} = \frac{2 + i\omega\mu_0 c^+}{d(2 - d/c^+)},
\]

where according to (A5) \( 1/c^- = 1/c^+ + i\omega\mu_0 c^+ \). Eq. (A6) yields \( c^+ = c(z_2) + d \). Since \( \Re c(z_2) \geq 0 \), we have \( \Re c^+ \geq d \), which implies \( \Re(d/c^+) \leq 1 \). Moreover, \( \mathcal{J}_{\text{in}}(d/c^+) > 0 \). Therefore the real parts of the numerator and denominator in the last fraction of (25) are of the same sign, and the imaginary parts are of opposite sign. Hence the compatibility condition cannot be satisfied.

A mobile sheet at \( \zeta \in (z_1, z_2) \) can exist, however, in the extremal models for \( \sigma_{\text{min}} \). Assuming a sheet at \( z_2 \), the compatibility condition \( \mathcal{J}_{\text{in}}[F'(\zeta)/F(z_2)] = 0 \) resulting from the system

\[
\Re{\lambda F'(\zeta)} = -1/\Delta,
\]

\[
\Re{\lambda F'(\zeta)} = 0,
\]

\[
\Re{\lambda F(z_2)} = 0
\]

can be satisfied. Examples are given in Section 3.2.3. However, a second mobile sheet at \( z = \zeta' \) cannot arise. Without restricting generality, let \( \zeta' > \zeta \). The relevant equations are then

\[
\Re{\lambda F'(\zeta')} = -1/\Delta,
\]

\[
\Re{\lambda F'(\zeta')} = 0,
\]

\[
\Re{\lambda F(z_2)} = 0.
\]

The subsystem consisting of the first three equations has no solution, since it agrees with the incompatible system for \( \sigma_{\text{max}} \) (with \( \zeta' \) and \( -1/\Delta \) replaced by \( z_2 \) and \( +1/\Delta \), respectively).

Proof of Proposition 3

This proposition is proved first for \( \sigma_{\text{max}} \). Assume at the outset that the terminating perfectly conducting sheet is missing, and that the deepest finite-conductance sheet lies at \( z = \zeta \). According to Proposition 1 we have \( \zeta \leq z_2 \). Since \( F(z) = \text{constant} \) for \( z > \zeta \), the necessary condition \( D(z) = 0 \) for \( \zeta < z_2 \) or \( D(z_2) = 0 \) for \( \zeta = z_2 \) leads to \( D(z_2) = 1/\Delta \). As a consequence, \( \sigma_{\text{max}} \) would be highly and uniformly sensitive to conductivity changes when \( z > z_2 \), a behaviour which can be excluded. Therefore a perfectly conducting sheet at \( z = \zeta_{\infty} \) has to be assumed, and \( D(z) \) in \( z_2 < z < \zeta_{\infty} \) is given by \( D(z) = \mathcal{A}(\zeta_{\infty} - z)^2 \) with \( D(z_2) \geq 1/\Delta \). If the sheet at \( z = z_2 \) were absent, \( D(z) \) would simply decrease by \( 1/\Delta \) when passing \( z = z_2 \) from below, but would continue to increase when moving upwards such that it would be impossible to satisfy \( D(z) = 0 \) at the first conducting sheet encountered at \( z = \zeta \). A sheet at \( z_2 \), however, changes the slope and curvature of \( D(z) \) and fulfils the condition \( D(z) = 0 \) with \( \zeta' = z_1 \) or \( \zeta' = \zeta_{\infty} \). The sheet at \( z_2 \) still contributes to \( \sigma_{\text{max}}(z_1, z_2) \).

If, on the other hand, \( \sigma_{\text{min}} \) were to terminate with a perfectly conducting sheet at \( z = \zeta_{\infty} \), then \( D(z) = \mathcal{A}(\zeta_{\infty} - z)^2 \), \( A \geq 0 \) for \( z \in (z_2, \zeta_{\infty}) \). Let \( \zeta \) be the position of the deepest finite-conductance sheet. If \( \zeta < z_2 \) (i.e. if the sheet at \( z = z_2 \) is absent), then \( D(z) = \mathcal{A}(\zeta_{\infty} - \zeta)^2 + 1/\Delta > 0 \) and the necessary condition \( D(z) = 0 \) cannot be satisfied. If \( \zeta = z_2 \), then \( \Re{\lambda F(z_2)} = 0 \) (implying \( A = 0 \)) and according to (24) \( \mathcal{J}_{\text{in}}[F'(\zeta)] = 0 \). Also, \( \zeta = \zeta_{\infty} \), \( D(z) = \mathcal{A}(\zeta_{\infty} - z)^2 \), and \( \sigma_{\text{min}} \) would be independent of the data, which only holds in exceptional cases (see, for example, the model for Region B in Section 3.2.3). Therefore the models of \( \sigma_{\text{min}} \) terminate with an insulator, implying \( D(z) = 0 \) for \( z > z_2 \). If the sheet at \( z = z_2 \) is absent, \( D(z) \) would merely increase by \( 1/\Delta \) when passing \( z = z_2 \) from below, and the condition \( D(z) = 0 \) could again not be satisfied. The remedy is a conducting sheet at \( z = z_2 \). It does not contribute to \( \sigma_{\text{min}}(z_1, z_2) \).

3.2.2 The extremal models for \( \sigma_{\text{max}} \)

To start with, we assume exact data such that \( c = c[\sigma] \). A secure point of departure and arrival is the representation of \( c \) by a two-parameter model with a surface sheet of conductance \( \tau(0) \) and a perfectly conducting sheet at \( \zeta_{\infty} \):

\[
c = c[\sigma] = \frac{\zeta_{\infty}}{1 + i\omega\mu_0 \tau(0)\zeta_{\infty}},
\]

with

\[
\zeta_{\infty} = \frac{|c|^2}{g}.
\]
It was shown in W1 and by Yee & Paulson (1988) that this model has two extremal properties, namely the greatest surface conductance and the shallowest perfect conductor of all models fitting $c$. Hence it is the starting point when $z_2$ is small. On the other hand, $\sigma_{\text{max}}$ will become infinite when $z_2 > \zeta_m = z_2M$, whereas it remains finite for $z_1 < z_2 < \zeta_m$.

In what follows, the structure of the extremal models and the corresponding extremal averages are displayed for all possible combinations $z_1 < z_2$ in triangular diagrams, for which Fig. 1 is the first example. The data are representative of the c-response of the diurnal $S_q$-harmonic. The diagram, being drawn to scale. All models terminate with a sheet of infinite conductance.

The control function $D(z)$, which plays a key role in the model construction, is displayed in Fig. 2 for four pairs $(z_1, z_2)$ from each of the regions. Before entering into a detailed description of the model-finding process, a discussion of this figure might be useful. Since $\sigma_+ = \infty$, the extremal conductivity average $Q_0 = -\sigma_{\text{max}}(z_1, z_2)$ is insensitive to small changes of the upper conductivity bound, i.e. according to (10) and (15) $\mu_+(z) = 0$ and $D(z) = \mu_-(z)$. If, in the small range from $z = -\delta z/2$ to $z + \delta z/2$, the lower bound $\sigma_-$ is perturbed from 0 to $\sigma_0$, and if $\delta \tau_-(z) = \sigma_0 \delta z$, then in the limit $\delta z \to 0$ (10) yields

$$D(z) = -\frac{\partial \sigma_{\text{max}}(z_1, z_2)}{\partial \tau_-(z)}.$$  

Consider first a (positive) perturbation $\delta \tau_-(z)$ at $z \in (z_2, \zeta_m)$ where the lower bound is active, $\sigma(z) = 0$. To restore the fit to the data, this perturbation has to be compensated by a (smaller) decrease in the conductance $\tau(z)$ at the shallower level $z_1$. Therefore, according to (27), $0 < \Delta D(z) < 1$. If $z \to z_2^+$, the perturbation $\delta \tau_-(z_2^+)$ is simply balanced by decreasing the adjacent conductance $\tau(z_2)$ by $\delta \tau_-(z_2)$, and thus decreasing $\sigma_{\text{max}}(z_2, z_2)$ by $\delta \tau_+(z_2)$. Therefore, according to (27), $\Delta D(z_2^+) = 1$. On the other hand, a perturbation of the lower bound at $z_2^-$ by $\delta \tau_-(z_2^-)$ is again compensated by a corresponding decrease of $\tau(z_2)$, but now the total conductance in the range $[z_1, z_2]$ remains unchanged and therefore $D(z_2^-) = 0$. The conductance increase due to a perturbation $\delta \tau_-(z)$ at a point $z \in (z_1, z_2)$ is also compensated by a decrease in $\tau(z_2)$. Since $z_2 > z$, the skin effect requires that the decrease at $z_2$ is stronger than the increase at $z$, leaving a net decrease of $\sigma_{\text{max}}(z_1, z_2)$, i.e. $D(z) > 0$. If the sheet at $z_1$ is present, the discussion

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**Figure 1.** Structure of the unconstrained extremal models for $\sigma_{\text{max}}(z_1, z_2)$. The icons representing the position of the conducting sheets are not drawn to scale. All models terminate with a sheet of infinite conductance. $Q$ is the quadruple point, existing only for $g \leq k$.
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Figure 2. Typical graphs of the control function $D(z)$ for the regions A–D of Fig. 1. In all cases $\Delta = 350\text{km}$.

Concerning $D(z)$ near $z = z_2$ also pertains to $z = z_1$. If the sheet at $z_1$ is absent (regions A and D), the perturbations near $z = z_1$ are balanced by a stronger decrease of $\tau(z_2)$. For $z \to z_1^+$, the positive perturbation $\delta \tau(z_1^+)$ contributes to $\sigma_{\text{max}}(z_1, z_2)$ and partially counteracts the decrease of $\tau(z_2)$, whereas for $z \to z_1^-$, the perturbation is not taken into account in $\sigma_{\text{max}}(z_1, z_2)$, such that—at least for region A—it is changed reflects the full decrease of $\tau(z_2)$.

Region A
The requirements for an extremal model are that the two-sheet model with the two unknown parameters $\tau(z_2)$ and $\zeta_\infty$ fits the data and that the objective function $Q[\sigma]$ is insensitive to changes in the model parameters. Therefore, according to (3), (23), and (24) the relevant equations are

$$c[\tau] = c,$$
$$\Re[\lambda F(z_2)] = 1/\Delta,$$
$$\Im[\lambda F(z_2)] = 0,$$

i.e. four real equations for the two model parameters and the complex $\lambda$. The former are easily obtained from the first equation, whereas the two remaining equations serve to determine $\lambda$:

$$\lambda = \frac{1}{F(z_2)\Delta}.$$  \hspace{1cm} (28)

Since the sheet at $z_2$ and the final conductor are present in all models, (28) invariantly holds for all four regions; the value of $F(z_2)$, however, changes. Using (12) and $f(z) = c - z$, $0 \leq z \leq z_2$ in region A, we obtain

$$\lambda = -\frac{1}{i\rho_0(c - z_2)\Delta}.$$  \hspace{1cm} (29)

With the known model parameters and $\lambda$, the control function $D(z)$ is easily determined. An example is shown in Fig. 2(A). As a necessary extremal condition, $D(z)$ is non-negative. The discontinuities at $z_1$ and $z_2$ are due to $w(z) = -1/\Delta$ for $z \in (z_1, z_2)$, and $w(z) = 0$ elsewhere. Between sheets, $D(z)$ varies quadratically. Model A ceases to be valid as soon as a further increase of $z_2$ leads to the vanishing of the function $D(z)$ at $z = 0$ or $z = z_1$, signalling the need to introduce an additional sheet. Assuming that the first $D(z_1) = 0$ (see Fig. 2B), we enter region B.

Region B
We now need a three-sheet model, where the parameters $\tau(z_1)$, $\tau(z_2)$ and $\zeta_\infty$ satisfy

$$c[\tau] = c,$$
$$\Re[\lambda F(z_1)] = 1/\Delta,$$
$$\Re[\lambda F(z_2)] = 1/\Delta,$$
$$\Im[\lambda F(z_2)] = 0.$$

The last three equations, considered as a linear system for $\lambda$, require the compatibility condition

$$\Re[\lambda F(z_1)/\lambda F(z_2)] = 1,$$  \hspace{1cm} (30)

which along with the first (complex) equation determines the three model parameters. $\lambda$ is then given by (28). Since $\tau(z_1)$ vanishes at the boundary between regions A and B, $f(z) = c - z$, $0 \leq z \leq z_2$, still applies. Therefore the boundary A–B is defined by

$$\Re\left(\frac{c - z_1}{c - z_2}\right)^2 = 1.$$  \hspace{1cm} (31)

After discarding the solution $z_1 = z_2$, this is an algebraic equation of first order in $z_1$ and third order in $z_2$, which is easily solved (see Appendix B.1).

Region D
If in model A, $D(z)$ first vanishes at $z = 0$ when $z_2$ is increased, we pass from region A to region D (see Fig. 2D). The relevant
equations are
\[ c[\sigma] = c, \]
\[ \mathcal{R}[\lambda F(0)] = 0, \]
\[ \mathcal{R}[\lambda F(z_1)] = 1/\Delta, \]
\[ \mathcal{F}_m[\lambda F(z_2)] = 0. \]

The two homogeneous equations in \( \lambda \) are compatible only if
\[ \mathcal{R}[F(z_2)/F(0)] = 0 \quad \text{or} \quad \arg(f(z_2)/f(0)) = -\pi/4, \] (32)
where we have discarded \(+\pi/4\) because the phase only decreases with depth. Eq. (32) implies that in extremal models requiring a surface sheet, the electric field at \( z_2 \) lags 45° behind its surface value. Eq. (32) and \( c[\sigma] = c \) enable the computation of the model parameters \( \tau(0), \tau(z_2) \) and \( \xi_{\infty}. \)

At the boundary A-D, where \( \tau(0) \) still vanishes, (32) reduces to
\[ \arg(1 - z_2/c) = -\pi/4 \quad \text{or} \quad z_2 = |c|^2/(g + h). \] (33)

Since no sheet at \( z_1 \) is present, the boundary A-D does not depend on \( z_1 \).

The quadruple point \( Q \) lies at the intersection of the boundaries A-B and A-D. After introducing (33), eq. (31) reduces to
\[ u_{\text{max}} = \frac{h|c|^2}{(g + h)}, \quad z_2 = \frac{|c|^2}{g + h}. \]

Since \( z_1 \leq z_2 \), a quadruple point exists only for \( h \leq g \), i.e. for an impedance phase \( \varphi \geq 45° \). If \( g < h \), for any \( z_1 \), the pertinent models change continuously from A to D when increasing \( z_2 \) from 0 to \( z_{2\text{M}} \).

Region C

The small intermediate region C has the most complicated structure, since here the sheets at \( z = 0 \) and \( z = z_1 \) coexist (Fig. 2C). We have to solve the system of equations
\[ c[\sigma] = c, \]
\[ \mathcal{R}[\lambda F(0)] = 0, \]
\[ \mathcal{R}[\lambda F(z_1)] = 1/\Delta, \]
\[ \mathcal{R}[\lambda F(z_2)] = 1/\Delta, \]
\[ \mathcal{F}_m[\lambda F(z_2)] = 0, \]
where the last four linear equations in \( \lambda \) now require the simultaneous validity of (30) and (32), which in turn, along with \( c[\sigma] = c \), are used to determine the four model parameters.

At the boundary C-D, the sheet at \( z_1 \) is still absent. Therefore \( f(z) = c(1 - z/c^+) \) in \( 0 \leq z \leq z_2 \), where \( c^+ \) with \( 1/c^+ = 1/c - i\omega_0 \tau(0) \) is the response function at \( z = 0 \) above the surface sheet. Hence, the compatibility conditions at the boundary are
\[ \arg(1 - z_2/c) = -\pi/4, \]
\[ \mathcal{R}[1 - z_1/c^+]^2/(1 - z_2/c^+)^2] = 1. \]

Solving the first condition for \( \tau(0) \), namely \( \omega_0 \tau(0) = 1/z_{2\text{M}} - 1/z_2 \), and inserting the result in the latter, it is found that the boundary C-D is defined by the simple relation
\[ z_1 + z_2 = |c|^2/g = 2z_{2\text{M}}. \]

The same principle is followed for the determination of the boundary B-C, where the surface sheet is still missing. We have \( f(0) = c \) and \( f(z_1) = c - z_1 \), whereas \( f(z_2) \) has to be expressed in terms of the unknown conductance \( \tau(z_2) \). This is determined from (32) and then by inserting \( f(z_2) \) into (30), which finally determines the boundary. The explicit expression is given in Appendix B.1.

For the one-frequency data set of Fig. 1, isolines of \( \sigma_{\text{max}} \) are shown in Fig. 3 in a triangular presentation. As expected, \( \sigma_{\text{max}} \) tends to infinity both for \( z_1 \rightarrow z_2 \) and for \( z_2 \rightarrow z_{2\text{M}} \). In the first limit, any thin sheet at \( z_2 \) yields an unbounded average conductivity; in the second limit, \( z_2 \) approaches the shallowest perfect conductor compatible with the data. Of interest are those parts of the diagram where \( \sigma_{\text{max}} \) is small, because these averages are constrained by the data. In the present example, the average conductivity between the surface and 275 km is the most constrained: whatever 1-D model we take to fit this two-data set, none will have an average conductivity exceeding 72 mS m\(^{-1}\) between 0 and 275 km.

So far, only exact data have been considered. If errors are taken into account, according to (4) we have to satisfy
\[ c[\sigma] - c \leq s, \] (34)
where for simplicity the numerical \( \chi^2 \)-bound is absorbed into \( s \). At the new extremum, (34) is assumed to be satisfied as an equality, i.e. \( \Lambda > 0 \), since in the case of an inactive constraint, \( \Lambda = 0 \), the extremum would not depend on the \( \chi^2 \)-bound. Hence the extremal model satisfies
\[ c[\sigma] = c + \delta c, \quad |\delta c| = s. \]

Because of the simplicity of the one-frequency case, one can easily spot the new extremum through a line search on the periphery of the circle \( |\delta c| = s \).

If \( s/|c| \) is small, however, an approximate first-order determination of the new extremum using the Lagrangian multiplier \( \lambda \) is adequate. Recalling that \( Q_0 = -\sigma_{\text{max}} \) and \( c = g - i h \), eq. (9) yields
\[ \delta \sigma_{\text{max}} \simeq \mathcal{R}(\lambda \delta c), \]
which is maximized and minimized for \( \arg \delta c = -\arg \lambda \) and \( \arg \delta c = \pi - \arg \lambda \), respectively. If \( \delta \sigma_{\text{max}} \) is the extremum for exact data, the extremum for inexact data will therefore vary between approximately \( \delta \sigma_{\text{max}} - |\lambda|s \) and \( \delta \sigma_{\text{max}} + |\lambda|s \), where in the present context only the upper bound of \( \delta \sigma_{\text{max}} \) is relevant. Taking as an example \( \delta \sigma_{\text{max}} \) for region A, we have
\[ \delta \sigma_{\text{max}} = \frac{h}{\omega_0 \Delta |c - z_2|^2}. \]

Therefore, using \( \lambda \) from (29),
\[ \frac{h - s}{|c - z_2|^2} \leq \omega_0 \Delta \delta \sigma_{\text{max}} \leq \frac{h + s}{|c - z_2|^2}. \] (35)
The exact determination by a line search—assuming that \( s \) is so small that all points of the periphery lie in region A—provides the slightly asymmetric bounds
\[ \frac{h - s}{|c - z_2|^2} \leq \omega_0 \Delta \delta \sigma_{\text{max}} \leq \frac{h + s}{|c - z_2|^2 - s^2}, \] (36)
which to first order in \( s/|c - z_2| \) agree with (35). It is noted, however, that in region A (35) underestimates the relevant upper bound. Even (36) underestimates this bound if the peripheral maximum point lies outside region A. The appropriate
method for noisy data in the multifrequency case is described in Section 4.1.4.

Explicit expressions for all parameters of the extremal models in regions A–D and further relevant material is collected in Appendix B.1.

3.2.3 The extremal models for $\sigma_{\text{min}}$

Now the conducting material has to be distributed in such a way that the data are satisfied and the average conductivity between $Z_1$ and $Z_2$ is as small as possible. A major difference to the minimization of the conductance between the surface level and $Z_2$, as considered in W1, is the fact that a possible surface sheet has to be included in the conductance, but is excluded from $\sigma_{\text{min}}$, since one may consider $Z_1 = 0^+$. Therefore the minimal conductance model is not a point of departure for $Z_1 \to 0$.

Generally, whenever a sheet is required in a $\sigma_{\text{min}}$-model at the end points of the range of integration, it lies at $Z_1$ or $Z_2$ and does not contribute to $\sigma_{\text{min}}$. Therefore in the simple one-frequency problem, for a wide range of parameters $(Z_1, Z_2)$, conductor configurations exist which yield $\sigma_{\text{min}}(Z_1, Z_2) = 0$. Fig. 4 shows the possible situations for $\sigma_{\text{min}}$. Again, four regions are required to cover the full $(Z_1, Z_2)$-space. Here, A and B lead to a vanishing conductivity average, whereas C and D show the anticipated mobile sheet of conductance $\tau$ at $Z \in (Z_1, Z_2)$. Representative graphs of $D(Z)$ are displayed in Fig. 5. A physical interpretation of $D(Z)$ as the sensitivity of $\sigma_{\text{min}}$ to a perturbation of the lower bound $\sigma_-$ at depth $Z$,

$$D(Z) = \frac{\partial \sigma_{\text{min}}(Z_1, Z_2)}{\partial \sigma_-(Z)}$$

[cf. also (27)], is again rewarding. For instance, the perturbation of the data due to a positive perturbation of this bound near $Z = Z$ can be compensated by a corresponding decrease of $\tau$, such that $\sigma_{\text{min}}(Z_1, Z_2)$ remains unchanged, and therefore $D(\tau) = 0$. On the other hand, a positive perturbation at $Z \in (Z_1, Z_2)$ is balanced by a reduction of both $\tau$ and $\sigma_-(Z_2)$, and therefore leads to a net increase of $\sigma_{\text{min}}(Z_1, Z_2)$, since the sheet at $Z_2$ does not contribute to $\sigma_{\text{min}}$. Perturbations in $Z > Z_2$ are fully compensated by $\tau(Z_2)$ and hence do not affect $\sigma_{\text{min}}$.

Region A

The data $c$ can be interpreted by a two-parameter model consisting of a single sheet of conductance $\tau$ at depth $Z$: 

$$c = g - ih = \zeta + \frac{1}{i \omega \mu_0 \lambda}.$$  

(37)

Therefore $\sigma_{\text{min}}(Z_1, Z_2) = 0$ whenever $(Z_1, Z_2)$ lies below this sheet, $g < Z_1 < Z_2$. In this region, $\sigma_{\text{min}}$ does not depend on infinitesimal changes of the data. Hence, $\lambda = 0$ and $D(Z) = w(Z)$. Since $w(Z) \geq 0$ according to (6), the extremal condition (16) is satisfied with
Figure 4. Structure of the unconstrained extremal models for $\delta_{\text{min}}(z_1, z_2)$. For $z_1 \to \infty$ the boundary B–C approaches the dashed line $z_1 = g$. Region D exists only for $g \geq h$. The models giving $\delta_{\text{min}}(z_1, z_2) = 0$ are non-unique.

$D(z) = 0$ for $z \notin (z_1, z_2)$. This is a trivial example where equality holds in (16) although $\sigma(z)$ attains its lower bound.

**Region B**

The data $c$ can also be interpreted by a three-parameter model consisting of a thin sheet of conductance $\tau$ at $z = \zeta < g$ and a perfectly conducting sheet at $C_\infty$:

$$\frac{1}{c - \zeta} = \sigma_0 \tau + \frac{1}{\zeta_\infty - \zeta},$$

where, taking the real part, $\zeta$ and $\zeta_\infty$ are related by $\zeta_\infty = g + \frac{h^2}{g - \zeta}$. Therefore $\delta_{\text{min}}(z_1, z_2) = 0$ also if $(z_1, z_2) \in (\zeta, \zeta_\infty)$. Here again $D(z) = w(z)$.

In regions A and B the conductivity $\sigma(z)$ is not completely specified for $z \notin (z_1, z_2)$, except for the modest requirement that it should fit the data. However, the extremal models become unique and $\delta_{\text{min}} > 0$, when $(z_1, z_2)$ can no longer be accommodated in $(\zeta, \zeta_\infty)$. Therefore the boundary between regions B and C in Fig. 4 is defined by $z_1 = \zeta$, $z_2 = \zeta_\infty$, i.e.

$$z_2 = g + \frac{h^2}{g - z_1} = z_1 + \frac{|c - z_1|^2}{g - z_1}.$$  \hspace{1cm} (38)

**Region C**

Now a mobile sheet of conductance $\tau$ at $z = \zeta \in (z_1, z_2)$ is required. The relevant system of equations is

$$c[\zeta] = c,$$

$$\mathfrak{Re}[\lambda F(z_1)] = 0,$$

$$\mathfrak{Re}[\lambda F(\zeta)] = -1/\Lambda,$$

$$\mathfrak{Re}[\lambda F(z_2)] = 0,$$

$$\mathfrak{Re}[\lambda F(\zeta_\infty)] = 0.$$  \hspace{1cm} (39)

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Figure 5. Typical graphs of \(D(z)\) for the regions A–D of Fig. 4. In panels A and B we have \(D(z) = w(z)\), i.e. the extremum is insensitive to small changes in the data.

which accounts for the fact that the sheets actually lie at \(z_1\) and \(z_2\), where \(w(z) = 0\). The homogeneous equations require the two compatibility conditions

\[
\mathcal{J}_y[F(\zeta)/F(z_1)] = 0, \quad \mathcal{J}_y[F(\zeta)/F(z_2)] = 0,
\]

which, along with (39), form a system of four non-linear equations for the model parameters \(\tau(z_1), \tau, \zeta, \text{ and } \tau(z_2)\). The closed-form solution is given in Appendix B.2. The Lagrangian multiplier is

\[
\lambda = \frac{i}{F(z_1) \mathcal{J}_y[F(\zeta)/F(z_2)]} \Delta.
\]

Region D

The conductance \(\tau(z_1)\) decreases steeply. The approximation given above leads to

\[
\tau(z_1) \approx \frac{(3/4)\sqrt{3}\varepsilon} {w_0 \Delta} \frac{[(c - z_1)^2]/(c - z_1)^2]}{1 - 9(c - z_1)(z_2 - z)/(2\Delta^2)},
\]

\[
\Delta D(z) \approx \begin{cases} 1, & 0 \leq z \leq z_1, \\ 1 - 3(z_2 - z)/(2\Delta), & \zeta \leq z < z_2, \\ 0, & z \geq z_2, \end{cases}
\]

which is different from \(D(z) = w(z)\) valid for \(\varepsilon = 0\). As required, \(D(z) \geq 0, D(\zeta) = 0, D' = 0, \text{ and } D(z_1) = D(z_2) = 1/\Delta\). It is remarkable that \(D(z)\) is only linear in \(\zeta \leq z < z_2\).

Finally, isolines of \(\sigma_{min}\) for the data considered previously for \(\sigma_{max}\) are shown in Fig. 6. As expected, \(\sigma_{min}\) is poorly constrained by this small data set. The greatest bound with \(\sigma_{min}(z_1, z_2) = 33.4 \text{ mS m}^{-1}\) can be obtained for the average conductivity between \(z_1 = 275 \text{ km}\) and \(z_2 = 1057 \text{ km}\), because these limits are sufficiently above and below \(g\) (the 'centre of gravity' of induced currents), such that the conducting material cannot be concealed. The maximum value of \(\sigma_{min}\) is of the order of the apparent conductivity \(\sigma_a = 1/\rho_a = 29 \text{ mS m}^{-1}\).

Data errors can be taken into account in the same way as described for \(\sigma_{max}\), i.e. either in the linear approximation on using the Lagrangian multiplier \(\lambda\) or exactly by a line search on the periphery of the circle of radius \(s\) around \(c\).
The one-frequency extremal models already allow an approximate treatment of the multifrequency case simply by selecting the frequency that imposes the hardest constraint on the conductivity average for a given \((z_1, z_2)\)-range. If \(\bar{t}(z_1, z_2, \omega_j)\) is the average for frequency \(\omega_j\), then conservative estimates of the true extremals are defined by
\[
\bar{t}_{\text{max}}(z_1, z_2) := \min_j \bar{t}_{\text{max}}(z_1, z_2, \omega_j) \geq \bar{t}_{\text{max}}(z_1, z_2),
\]
\[
\bar{t}_{\text{min}}(z_1, z_2) := \max_j \bar{t}_{\text{min}}(z_1, z_2, \omega_j) \leq \bar{t}_{\text{min}}(z_1, z_2).
\]

A simple example is given in Fig. 7, which shows \(\tilde{\sigma}_{\text{max}}(z_1, z_2)\) for European average responses of the first six \(S_q\)-harmonics, based on the data of Olsen (1994). Each harmonic constrains the conductivity in a particular depth range (separated by dashed lines). The plot is constructed from the 1-D consistent \(D^*\)-data (Parker 1980), which essentially provide the same results as the measured data, but give a slightly clearer visualization of the depth sensitivity of the response estimates. All data are displayed in Table 1. The \(D^*\)-model has a 6.8 kS sheet at \(z = 170\) km, a 61.6 kS sheet at \(z = 603\) km, and an infinitely conducting sheet at \(z = 695\) km. As an example, for \(z_1 = 100\) km, and \(z_2 = 200\) km this model leads to a conductivity average \(\bar{\sigma}(z_1, z_2) = 68 \text{ mS m}^{-1}\), which suggests that the conservative upper bound of 80 mS m\(^{-1}\) read from Fig. 7 is not too pessimistic.

### 3.3 Constrained one-frequency extremal models

#### 3.3.1 General structure

So far \(\sigma(z)\) has been allowed to vary between 0 and \(\infty\). Now the constraints \(0 < \sigma_- \leq \sigma(z) \leq \sigma_+ < \infty\) are imposed, which increases \(\sigma_{\text{min}}\) and decreases \(\sigma_{\text{max}}\). It was seen in Section 3.1 that in the one-frequency case \(\sigma(z)\) only attains the values \(\sigma_-\) and \(\sigma_+\). The bounds \(\sigma\pm\) have to be chosen sufficiently wide to warrant the existence of a model satisfying the data (see Section 3.3.2). The resulting extremal models will not then differ too much from the unconstrained case: essentially, thin sheets will be replaced by layers of finite thickness and conductivity \(\sigma_+\), and insulators change into conducting layers with conductivity \(\sigma_-\). Again, a layer with conductivity \(\sigma_+\) lies immediately above \(z_2\) for \(\sigma_{\text{max}}\) and immediately below that level for \(\sigma_{\text{min}}\). According to (16) \(D(z) \geq 0\) for \(\sigma(z) = \sigma_-\), and \(D(z) \leq 0\) for \(\sigma(z) = \sigma_+\).

A major qualitative change of the model structure, however, occurs at greater depth, corresponding in the unconstrained
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Cleaned Sq-Harmonics

\( T = 4 - 24 \) h

\( \sigma_{\text{max}} \) [mS/m]

\[ \overline{\sigma}_{\text{max}}(z_1,z_2) \]

Figure 7. Isolines of the approximate multifrequency extremal \( \overline{\sigma}_{\text{max}}(z_1,z_2) \) for European averages of the first six Sq-harmonics, based on the data of Olsen (1994). The dashed lines delimit the regions, in which the conductivity is constrained by the harmonics given along the diagonal.

Table 1. \( g \)-Responses of the first Sq-harmonics from Olsen (1994). The radius of the one standard deviation error circle is \( s \). Fig. 7 is constructed from the 'cleaned' data (\( D^+ \)-approximation).

<table>
<thead>
<tr>
<th>( j )</th>
<th>( T )</th>
<th>( g )</th>
<th>( h )</th>
<th>( s )</th>
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<td>271</td>
<td>199</td>
<td>16</td>
<td>283</td>
<td>192</td>
</tr>
</tbody>
</table>

Table 1. \( c \)-Responses of the first Sq-harmonics from Olsen (1994).

\( c(\text{measured}) \) \( \text{std} \) \( c(\text{cleaned}) \)

\begin{align*}
\frac{f(\zeta_{i+1})}{f(\zeta_i)} &= q_+ = \frac{k_+ + 1/c_+}{k_+ + 1/c_-} \exp(-k_- d_-) \\
\frac{f(\zeta_{i+1})}{f(\zeta_i)} &= q_- = \frac{k_- + 1/c_-}{k_- + 1/c_+} \exp(-k_+ d_+) 
\end{align*}

where periodicity demands that

\( k_- c_+ + t_- = \frac{1}{1 + k_- c_+ t_-} \)

\( k_+ c_- + t_+ = \frac{1}{1 + k_+ c_- t_+} \)

\( t_+ = \tanh(k_+ d_+) \)

It is now necessary to solve these equations and \( \arg q_\pm = -\pi/2 \)

case to \( z > \zeta_{\text{max}} \) for \( \sigma_{\text{max}} \) and \( z > z_2 \) for \( \sigma_{\text{min}} \). Since \( E_z(z) \) performs a damped oscillation, the same applies to \( F(z) \) and \( D(z) \), such that all extremal models terminate with an infinite sequence of layers with alternating conductivities of \( \sigma_- \) and \( \sigma_+ \). It is now shown that the necessary extremal conditions are satisfied by a periodically layered half-space with layer thicknesses of \( \pi/2 \) times the local penetration depth. For this purpose let \( \zeta_i \) and \( \zeta_{i+1} \), with \( z_2 < \zeta_i < \zeta_{i+1} \), be the positions of two consecutive layer boundaries. Then (16) requires that

\( \mathcal{R}[\lambda F(\zeta_i)] = 0, \quad \mathcal{R}[\lambda F(\zeta_{i+1})] = 0, \)

implying that

\( \mathcal{R}[f(\zeta_{i+1})/f(\zeta_i)] = \mathcal{R}[f^2(\zeta_{i+1})/f^2(\zeta_i)] = 0 \).

To avoid additional zeroes of \( D(z) \) between \( \zeta_i \) and \( \zeta_{i+1} \), we have to choose the smallest possible phase shift between \( f(\zeta_i) \) and \( f(\zeta_{i+1}) \). Hence, \( \arg[f(\zeta_{i+1})/f(\zeta_i)] = -\pi/2 \). Let \( k_\pm = \sqrt{\mu_\infty \sigma_\pm} \) and let \( c_\pm \) be the response function on the top of a layer with conductivity \( \sigma_\pm \) and thickness \( d_\pm \). Depending on whether the conductivity between \( \zeta_i \) and \( \zeta_{i+1} \) is \( \sigma_- \) or \( \sigma_+ \), the field variation between the bottom and top is easily found to be

\[ f(\zeta_{i+1})/f(\zeta_i) = q_+ = \frac{k_+ + 1/c_+}{k_+ + 1/c_-} \exp(-k_- d_-) \]

or

\[ f(\zeta_{i+1})/f(\zeta_i) = q_- = \frac{k_- + 1/c_-}{k_- + 1/c_+} \exp(-k_+ d_+) \]
for \(d_\pm\) and \(c_\pm\). The result is
\[
k_\pm d_\pm = k_\pm c_\pm = (1 + i)\pi/2 \quad \text{and} \quad c_\pm = 1/\sqrt{\text{lozenge}_0}\sigma_\pm. \quad (46)
\]
with
\[
\sigma_+ / \sigma_- = [\gamma (1 - r) + \sqrt{2(1 - r^2 + 4r)^2}/4 > 1
\]

where
\[
g := \tan (\pi/2) = 1.09033 \quad \text{and} \quad r := \sqrt{\sigma_- / \sigma_+}.
\]

The layers have thickness \(d_\pm = (\pi/2)p_\pm\), where \(p_\pm := \sqrt{2/(\text{lozenge}_0}\sigma_\pm)\) is the local penetration depth.

On the top of a layer with \(\sigma = \sigma_-\) or \(\sigma_+\), the sequence is equivalent to a uniform half-space with apparent conductivity \(\sigma_-\) or \(\sigma_+\), which are, respectively, smaller or greater than the smallest or greatest conductivity of the constituents forming the conductor. Physically this is obvious from the fact that the periodic structure forms a sequence of quarter-wavelength plates (wavelength 2\(\pi p_\pm\)) with a phase lag of \(\pi/2\) for each traverse of a layer. At an interface, the phase lag of the upward propagating reflected wave with respect to the incident field is always a multiple of \(\pi\): the reflected field has twice traversed all layers above the deepest point of reflection and has undergone additional phase shifts of \(\pi\) at interfaces to good conductors. In particular, due to the dominant contribution of the reflection at the first interface, the reflected field is in-phase with the incident electric field at the top of a layer with \(\sigma = \sigma_-\) (thus increasing \(|E|\)) and has a phase shift \(\pi\) at the top of a layer with high conductivity. Therefore \(\sigma_- < \sigma_+\) and \(\sigma_{++} > \sigma_+\).

### 3.3.3 Data constraints

The necessary and sufficient conditions that one-frequency data \(c\) can be interpreted by a 1-D model are \(g \geq 0, h \geq 0\), corresponding to the phase constraints \(0 \leq \varphi \leq \pi/2\). However, if \(c\) is to be interpreted by an element of the constrained conductivity set, additional conditions have to be imposed on the data. This is obvious in the extreme case \(\sigma_- = \sigma_+\), which allows only uniform half-spaces and subjects \(c\) to the strict requirements that \(\sigma_0 = \sigma_\pm\), \(g = \pi/4\). The dependence of the data constraints on \(\sigma_-\) and \(\sigma_+\) will now be investigated. For this purpose we assume exact data, select \(\sigma_-\) and \(\sigma_+\), and determine the extremal phase values at which a constrained conductivity model no longer exists for a given \(\sigma_+\). The procedure is similar to the construction of extremal conductivity averages and is outlined only briefly. The Lagrange function corresponding to (7) is
\[
L[\sigma] = \varphi[\sigma] + \kappa[\sigma_0[\sigma] - \sigma_0]
\]
\[
+ \int_0^\infty \left[ v_+(z) [\sigma(z) - \sigma_+] + v_-(z) [\sigma_- - \sigma(z)] \right] dz
\]
where \(\kappa\) and \(v_\pm(z) \geq 0\) are the Lagrangian multipliers. A necessary extremal condition is that the first variation with respect to \(\sigma\) has to vanish, yielding from
\[
\frac{\partial c}{c} = - \frac{\delta \sigma_0}{2\sigma_0} + i\delta \varphi
\]
and (11) that
\[
G(z) = \beta \left[ F(z)/c - 2\sigma_0 \delta c [F(z)/c] = v_+(z) - v_-(z). \right.
\]
\[G(z)\] corresponds to the control function \(D(z)\) and satisfies conditions analogous to (16). Again, \(G(z)\) cannot vanish in a finite interval, because the simultaneous validity of \(G(z) = 0\) and \(G'(z) = 0\) would imply that \(\beta \left[ F(z)/c \right] = 0\), whereas this value is in fact \(-2h(z)(\epsilon z)^2 < 0\). Therefore the extremal phase models also consist of a stack of layers with alternating conductivities \(\sigma_-\) and \(\sigma_+\), and \(G(z) = 0\) at all internal interfaces \(z = z_i, i = 0, 1, \ldots, \text{and} z_i < z_{i+1}\). Then \(G(z) = 0\) and \(G(z) = 0\) imply that
\[
\beta \left[ F(z_{i+1})/F(z_i) \right] = \frac{\beta \left[ F(z_{i+1})/F(z_i) \right]}{\beta \left[ F(z_{i+1})/F(z_i) \right]} = 0,
\]
\(i = 0, 1, \ldots\). This condition requires that \(\arg \left[ F(z_{i+1})/F(z_i) \right] = \pi/2\). Since the phase decreases with depth, this phase shift is negative, and, since two neighbouring zeros are considered, \(\arg \left[ F(z_{i+1})/F(z_i) \right] = -\pi/2\). The previous section showed that this condition can be satisfied by a periodic sequence of quarter-wavelength plates in \(z \geq z_0\). The free parameter \(z_0\) is adjusted to yield the assigned apparent conductivity \(\sigma_+\).

The construction of the extremal phase models is now obvious: since small phase values less than \(45^\circ\) occur if the conductivity decreases with depth, the minimum phase model is obtained by considering a periodic sequence of quarter-wavelength plates with \(\sigma_-\) in the top layer, overlain by a layer of conductivity \(\sigma_+\) and variable thickness \(z_0, 0 \leq z_0 \leq d_+\). The response \(c(z_0)\) is then given by
\[
c(z_0) = \frac{1}{k_+ c_+ + t}
\]
where \(c_+\) is defined by (46). Particular values are \(\sigma_0(0) = \sigma_- < \sigma_+, \sigma_0(0.5d_+) = \sigma_+, \sigma_0(d_+ = \sigma_0 > \sigma_+\). The \(\varphi\) and \(\varphi < 45^\circ\) in between. Similarly, the maximum phase model is obtained if the conductivity increases with depth, i.e. by placing a layer with conductivity \(\sigma_-\) and variable thickness \(z_0, 0 \leq z_0 \leq d_-\) over a periodic sequence with \(\sigma_+\) in the top layer. Identical extremal conditions are found when extremizing the apparent conductivity for a pre-assigned phase. Therefore the convex domains contoured in the \((\sigma_0, \varphi)-\)plane by varying the thickness of the top layer define the data constraint area, which are shown in Fig. 8 for various ratios \(\sigma_-/\sigma_+\) (curv parameter). The dashed vertical lines mark the under- and over-shoot areas, where the apparent conductivity is smaller than \(\sigma_-\), or greater than \(\sigma_+\).

Adopting a slightly different point of view, a simple physical meaning can be assigned to the contours in Fig. 8: if apparent conductivity and phase are continuously monitored along a downward descent through the periodic quarter-wave length structure, then the contours are followed in a clockwise direction. An interface is crossed whenever the phase equals \(45^\circ\).

The extremal phases and apparent conductivities are very close to the extremal values encountered in a simple two-layer structure with the conductivities \(\sigma_-\) and \(\sigma_+\). For instance, for \(\sigma_-/\sigma_+ = 100\) the two-layer earth yields the maximum phase 77.42°, which is only slightly smaller than the extremal phase 78.00°. Also, the maximum two-layer apparent conductivity of 1.2447\(\sigma_+\) is very close to the extremal value of 1.24885\(\sigma_+\). For \(\varphi = 45^\circ\), however, the two-layer apparent conductivities \(\sigma_+\) differ significantly from the true extremal values of \(\sigma_+ = 1.15\sigma_+\) and \(\sigma_- = 0.87\sigma_-\).

### 3.3.3 Examples

As for the unconstrained case, we again consider the diurnal S\(p\)-harmonic response (see Figs 1–6), now subject to the
conductivity constraints \( \sigma_- = 0.01 \, \text{S} \, \text{m}^{-1}, \sigma_+ = 1 \, \text{S} \, \text{m}^{-1} \). Then \( \sigma_+ \sqrt{\sigma_- \sigma_+} = 0.29, \varphi = 63^\circ \), such that according to Fig. 8 an interpretation by a constrained conductivity model is clearly possible for \( \sigma_+ / \sigma_- = 100 \).

First we consider the extremal models for \( \delta_{\text{max}} \). The required conductor configurations are shown in Fig. 9, which has to be compared with the corresponding Fig. 1 for the unconstrained case. Both figures agree on the characteristic quadruple-point topology. The finite upper bound \( \sigma_+ \) on the conductivity, however, leads to major changes for those pairs \((z_1, z_2)\) for which \( \delta_{\text{max}}(z_1, z_2) \to \infty \) in the unconstrained case, i.e. for \( z_2 \to z_{3M} \) and \( z_1 \to z_2 \).

1. For \( \sigma_+ = \infty \) and \( z_2 \to z_{3M} \) the thin sheet at \( z = z_2 \) increases its conductance and coalesces with the perfectly conducting sheet at \( z = z_{3M} \) (see Appendix B.1). A similar coalescence also occurs for \( \sigma_- < \infty \). Here the conducting layer in \( z \leq z_2 \) merges for large values of \( z_2 \) with the shallowest conducting \( \lambda/4 \)-layer forming the substratum. In the present case, this occurs for \( z_2 \approx 740 \, \text{km} \). The electromagnetic field now penetrates the level of coalescence. This necessitates the introduction of three new conductivity configurations, which differ from the corresponding configurations at smaller \( z_2 \) by the merged substructure.

The thickness of the conducting \( \lambda/4 \)-layer is 232 km. All models obtained by a variation of \( z_2 \) in this layer clearly maximize \( \delta(z_1, z_2) \) such that, apart from the position of \( z_2 \), the extremal models remain unchanged for \( 740 \, \text{km} \leq z_2 \leq 972 \, \text{km} \). The merging process is illustrated by means of \( D(z) \) in Fig. 10(a)–(c). In the merged structure \( D(z_2) \) is so negative that the addition of \( 1/\lambda \) does not change the sign.

2. Owing to the finite value of \( \sigma_+ \), the thin sheet at \( z_2 \) is replaced by a finite layer in \( z \leq z_2 \). For a given \( z_2 \), the thickness of the layer in \( z \leq z_2 \) is given by the hatched area in Fig. 9. Whenever \((z_1, z_2)\) is in this area we obtain \( \delta_{\text{max}}(z_1, z_2) = \sigma_+ \). In this case \( \delta_{\text{max}} \) is insensitive to small data changes, i.e. \( \lambda = 0 \) and \( D(z) = w(z) \). An example is shown in Fig. 10(d). Again the necessary extremal conditions (16) are only weakly satisfied \( D(z) \leq 0 \) for \( \sigma(z) = \sigma_+ \) and \( D(z) = 0 \) for \( \sigma(z) = \sigma_- \).

The resulting isolines of \( \delta_{\text{max}}(z_1, z_2) \) are shown in Fig. 11, which should be compared with Fig. 3. The tightening of the bounds on \( \delta(z_1, z_2) \) by imposing constraints on \( \sigma(z) \) is clearly visible. The closest bound is obtained for the conductivity average between the surface and \( z_2 \approx 250 \, \text{km} \), which will not exceed 50 mS for any of the models satisfying this small data set.

The computation of the extremal models in the constrained case is very similar to that in the unconstrained case, although analytical solutions are no longer available. Apart from the degenerate case \( D(z) = w(z) \), i.e. \( \lambda = 0 \), interfaces occur whenever \( D(z) \) changes sign. At the discontinuities \( z = z_k \), \( k = 1, 2 \), this requires \( 0 < \Re(\lambda'F(z_k)) < 1/\lambda \). Examples considered are the computation of the quadruple point and the level of coalescence:

**Quadruple point.** As can be inferred from Fig. 9, the quadruple-point model consists of four layers, with \( \sigma = \sigma_- \) in \( 0 \leq z < \zeta_1; \sigma = \sigma_+ \) in \( \zeta_1 < z < \zeta_2; \sigma = \sigma_- \) in \( z_2 < z < \zeta_3 \); and \( \sigma = \sigma_+ \) in \( z > \zeta_3 \). Here \( \sigma_{\text{max}} > \sigma_+ \) represents the periodic substructure (see Section 3.3.1). (In the present example \( \sigma_{\text{max}} = 1.15 \, \text{S} \, \text{m}^{-1} \).) In addition to fitting the data, the quadruple-point coordinates \( z_2 \) and \( z_3 \) with \( 0 < z_1 < \zeta_3 \) have to be chosen in such a way that, by a small perturbation of the coordinates,
Figure 9. Structure of the constrained extremal models for \( \sigma_{\text{max}} \). The hatched parts of the icons mark layers with \( \sigma = \sigma_+ \). The three dots indicate the periodicity. The deep conducting layer already belongs (fully or in part) to the periodic quarter-wavelength substructure. The poorly conducting \( \lambda/4 \)-plate already has a thickness of 2324 km. For given \( z_1 \), the hatched area at the right marks the thickness of the conducting layer overlying \( z = z_2 \).

Figure 10. Typical graphs of \( D(z) \) for the model of Fig. 9. Panels (a)-(c) illustrate the coalescence of the conducting layer overlying \( z_2 \) with the first conducting layer of the substratum. Panel (d) pertains to a point \((z_1, z_2)\) of the hatched region of Fig. 9 where \( D(z) = w(z) \).
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$T = 24 \text{h}$
$c = (550 - 275i) \text{ km}$
$\sigma_\ominus = 10 \text{ mS/m}$
$\sigma_\oplus = 1000 \text{ mS/m}$
$
\bar{\sigma}_{\max}(z_1, z_2) [\text{mS/m}]
$

conducting layers at $z = 0$ and/or $z = z_1$ can emerge. This leads to the conditions

\[ c[\sigma] = c, \]
\[ \Re[\lambda F(0)] = 0, \]
\[ \Re[\lambda F(z_1)] = 1/\lambda, \]
\[ \Re[\lambda F(\zeta_1)] = 1/\lambda, \]
\[ \Re[\lambda F(z_2)] = 0. \]

The interface depth $z_2$ does not enter explicitly into this system, but is implicitly taken into account when calculating $F(z)$. Hence, the above system consists of six real equations for the six real unknowns $\Re \lambda$, $\Im \lambda$, $z_1$, $z_2$, $\zeta_1$, and $\zeta_2$. The linear Lagrangian multiplier $\lambda$ is again eliminated by the two compatibility conditions

\[ \Im F(z_2)/\Im F(0) = 0, \quad \Im [F(z_1) - F(\zeta_1)]/\Im F(0) = 0, \]

which along with the first (complex) equation form a system of four non-linear equations for the four position parameters.

**Level of coalescence.** The determination of this is considered for the simplest case where there is a conducting surface layer only. Thus the model again consists of four layers with $\sigma = \sigma_\ominus$ in $0 < z < \zeta_1$; $\sigma = \sigma_\ominus$ in $\zeta_1 < z < \zeta_2$; $\sigma = \sigma_\ominus$ in $\zeta_2 < z < z_2$; and $\sigma = \sigma_\oplus$ in $z > z_2$. The condition for coalescence at $z = z_2$ is $D(z_2^2) = 0$. Hence

\[ c[\sigma] = c, \]
\[ \Re[\lambda F(\zeta_1)] = 0, \]
\[ \Re[\lambda F(z_2)] = 1/\lambda, \]
\[ \Re[\lambda F(z_2)] = 0. \]

Therefore the position parameters $\zeta_1$, $\zeta_2$ and $z_2$ are determined from the first equation and the compatibility condition

\[ \Im [F(z_2)/F(\zeta_1)] = 0. \]

The solutions to the systems of non-linear equations are easily obtained by a continuous deformation of the unconstrained solution given in Appendix B.1 on using, for example, Brown's method (Brown 1973), which does not require the user to furnish the partial derivatives.

Now we briefly turn to the structure of the constrained extremal models for $\bar{\sigma}_{\min}(z_1, z_2)$, which is shown in Fig. 12. The models strongly resemble the unconstrained models (Fig. 4). A general feature is that the region in the $(z_1, z_2)$-plane where $\bar{\sigma}_{\min}$ exceeds $\sigma_\ominus$ has become broader (compare also Figs 6 and 13). A particularly complex sequence of conductivity models is required to transform the model with the mobile layer and the layer in $z < z_1$ (corresponding to model C in Fig. 4) to the final model for $z_1 = 0$, $z_2 \approx 580 \text{ km}$.

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where the two shallow conductors are missing and \( \bar{\sigma}_{\text{min}} = \sigma_{-} \). The layer in \( z < z_{1} \) touches the surface, then coalesces at \( z = z_{1} \) with the mobile layer, and the combined conductors then gradually vanish. This sequence of events (Fig. 12) happens in the range \( 0 \leq z_{1} \leq 3.2 \text{ km} \) only, but has been blown up for clarity. Finally, isolines of \( \bar{\sigma}_{\text{min}}(z_{1}, z_{2}) \) are displayed in Fig. 13. In comparison with Fig. 6 they show the desired increase of \( \bar{\sigma}_{\text{min}} \). \( \bar{\sigma}_{\text{min}} \) is most constrained between \( z_{1} \approx 400 \text{ km} \) and \( z_{2} \approx 1050 \text{ km} \), and always exceeds 53 mS.

The comparison of the isoline plots in Figs 3 and 11 for \( \sigma_{\text{max}} \) and in Figs 6 and 13 for \( \bar{\sigma}_{\text{min}} \) shows the possible tightening of average conductivity bounds by imposing \textit{a priori} information. Despite this improvement, one-frequency extremal models can provide only weak bounds on the underlying conductivity structure.

4 MULTIFREQUENCY EXTREMAL CONDUCTIVITY AVERAGES

4.1 Unconstrained conductivity models

4.1.1 A two-frequency example

The extremal models for one frequency switched between the extremal conductivities \( \sigma_{-} \) and \( \sigma_{+} \). In the multifrequency case \( (M > 1) \), this simple structure will also prevail, but the existence of continuous conductivity transitions can no longer be ruled out. Because of the complexity of the general case, the treatment for \( M > 1 \) must be confined to the presentation of partial results. First we will illustrate some of the new features through a two-frequency example.

Figure 12. Structure of the constrained extremal models for \( \bar{\sigma}_{\text{min}} \). In contrast to Fig. 9, the deep conducting layers are thinner than \( \lambda/4 \) and form no part of the quarter-wavelength structure, which starts with the 2324 km thick poorly conducting layer. The model structure near \( z_{1} \approx 650 \text{ km} \) has been enlarged in the \( z_{1} \)-direction.
Augmenting the previous one-frequency example by a shorter period, we consider the data set

\[ T_1 = 24 \text{ h}, \quad c_1 = (550 - 275i) \text{ km}, \]
\[ T_2 = 6 \text{ h}, \quad c_2 = (350 - 220i) \text{ km}, \]

which simulate the first and fourth Sq-harmonics. Following Yee & Paulson (1988), it is seen when constructing the lens-shaped consistency region for the response of period \( T_2 \), given the response for period \( T_1 \), that the data are optimally consistent (and not close to degeneration, i.e. \( c_2 \) is not close to the boundary of the consistency region).

The four real data can be mapped into two four-parameter thin-sheet models, which are the thin-sheet model with the greatest surface conductance and the shallowest perfect conductor (model I),

\[ \zeta_1 = 0 \text{ km}, \quad \tau_1 = 2592 \text{ S}, \]
\[ \zeta_2 = 460.3 \text{ km}, \quad \tau_2 = 33510 \text{ S}, \]
\[ \zeta_3 = 893.3 \text{ km}, \quad \tau_3 = \infty, \]

and the thin-sheet model exhibiting the deepest first conductor and smallest total conductance (model II),

\[ \zeta_1 = 214.7 \text{ km}, \quad \tau_1 = 8487 \text{ S}, \]
\[ \zeta_2 = 703.1 \text{ km}, \quad \tau_2 = 58179 \text{ S} \]

(see Weidelt 1985; Yee & Paulson 1988).

Let us consider in some detail the structure of the extremal models for \( \eta_{\text{max}}(\zeta_1, \zeta_2) \), which is shown in Fig. 14 for all pairs \((\zeta_1, \zeta_2)\). Although the increase of complexity compared with the one-frequency case (Fig. 1) is obvious, some simple properties are easily identified. Model I forms the backbone of the evolution of the models. Since it is the shallowest perfect conductor model, \( \eta_{\text{max}}(\zeta_1, \zeta_2) = \infty \) for \( \zeta_2 > \zeta_3 = 893 \text{ km} \) (bottom of the diagram). The level \( \zeta_2 = 460 \text{ km} \) is also important: for two frequencies most extremal models for \( \eta_{\text{max}} \) show a mobile sheet at \( z = \zeta_2 \) with \( \zeta > \zeta_2 \) for \( \zeta_2 < \zeta_2 \) and \( \zeta < \zeta_2 \) for \( \zeta_2 > \zeta_2 \). At \( \zeta_2 = \zeta_3 \), model I is realized for all \( \zeta_1 \). The sequence of events for an interchange \( \zeta > \zeta_2 \rightarrow \zeta < \zeta_2 \) is as follows. The mobile sheet coalesces for \( \zeta_2 \rightarrow \zeta_2^* \) with the perfect conductor...
at \( z = \zeta^+ \). Then for \( z_2 > \zeta^+ \) a new mobile sheet emerges in two ways: for small \( z_1 \) a sheet appears at \( z_1 \), which as \( z_2 \) increases starts moving downwards; for greater values of \( z_1 \), the surface sheet of model I becomes mobile.

The upper part \( z_2 \leq \zeta^+ \) of Fig. 14, in common with the one-frequency case of Fig. 1, has the three-partite sequence of models with a sheet at \( z_1 \), two sheets at \( z = 0 \) and \( z = z_1 \), and only one sheet at \( z = 0 \). The basic difference compared with Fig. 1 is that a quadruple point \( Q \) does not occur. (It can be imagined to lie outside the triangular diagram, as also happens in the one-frequency case for \( g < \eta \).) Moreover, the final perfect conductor disappears at \( z_2 = \zeta^+ \) for \( z_1 > 214 \) km (model II), but reappears for greater \( z_2 \).

The three-partite sequence also occurs in the lower part of the diagram for small \( z_1 \). The corresponding quadruple point now lies at \( z_1 = 150 \) km, \( z_2 \approx 660 \) km.

The most striking feature of Fig. 14 is the small shaded area, in which a continuous conductivity variation is required. The subsequent discussion refers to the fixed value \( z_2 = 750 \) km. At a mobile sheet \( z = \zeta \) the control function and its average slope vanish, i.e. \( D(\zeta) = 0, \overline{D}'(\zeta) = 0 \). In general, \( D'(\zeta^-) < 0 \) and \( D'(\zeta^+) > 0 \). In exceptional situations, however, one may find that \( D'(\zeta^-) = D'(\zeta^+) = 0 \). This situation occurs at the right border of the shaded area for \( z_1 = 468 \) km. Fig. 15 shows the graph of \( D(z) \) on the right (panel A). Since \( D(z) \) is a quadratic function, \( D(\zeta) = D'(\zeta) = 0 \) and \( D(0) = 0 \) imply that \( D(z) = 0 \) for \( z \leq \zeta \). If \( z_1 \) decreases, the slopes reverse their sign and small negative side lobes occur (panel B1 of Fig. 15), such that the first necessary condition (16) is violated. According to (27), \( D(z) < 0 \) indicates that \( \sigma_{max}(z_1, z_2) \) can still be increased by replacing the active bound \( \sigma = 0 \) by \( \sigma > 0 \) at regions with \( D(z) < 0 \), i.e. the model B1 for \( z_1 = 446 \) km is not extremal.

Therefore the extremal models for \( z_1 < 468 \) km are constructed by satisfying the second necessary condition of (16), \( D(z) = 0 \), by a continuous conductivity distribution \( 0 < \sigma(z) < \infty \) in \( 0 < z < \zeta \), where \( \zeta = \zeta(z_1) \) has to be determined. Hence, for \( z_1 = 446 \) km the wrong model B1 in Fig. 15 is replaced by model B2 which has a continuous variation in the hatched area. Thus the sheet with \( \zeta = 3316 \) S occurring at \( \zeta = 237 \) km for \( z_1 = 468 \) km is dissolved into a continuous profile for \( z_1 < 468 \) km. The hatched area gradually shrinks to zero when \( z_1 = 424 \) km at the left boundary of the shaded area in

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**Figure 14.** Structure of the extremal models \( \sigma_{max}(z_1, z_2) \) for a two-frequency problem. A continuous conductivity change is required in the shaded area.

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\[ T_1 = 24 \text{h}, c_1 = (550-275i) \text{km} \]
\[ T_2 = 6 \text{h}, c_2 = (350-220i) \text{km} \]
\[ \overline{\sigma}_{max}(z_1, z_2) \]
\[ \sigma^* = 0 \]
\[ \sigma^* = \infty \]
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Fig. 15. Typical graphs of $D(z)$ for the shaded area of Fig. 14 for variable $z_1$ and fixed $z_2 = 750$ km. The right boundary of the area (A) is defined by $D' = 0$ at a mobile sheet. The negative side lobes for smaller values of $z_1$ disqualify B1 as an extremal model. The correct model B2 shows a continuous conductivity variation in the hatched area, which gradually disappears when the left boundary is reached (C).

Fig. 14 is approached (see also panel C in Fig. 15). The resulting conductivity profiles are shown in Fig. 16 for a selection of $z_1$-values (curve parameter, in km).

The construction of these profiles is now briefly discussed. For this purpose (and further reference) we require the first derivatives of the Fréchet derivative $F(z)$. From (12) and (1) we obtain

\[ F'' = 2i\omega\mu_0\sigma F + (F')^2/(2F), \]  
\[ F' = 2i\omega\mu_0(2\sigma F' + \sigma' F) = 4i\omega\mu_0\sqrt{\sigma}(\sqrt{\sigma} F'), \]  
\[ F''' = 2i\omega\mu_0(2\sigma F'' + 3\sigma' F' + \sigma'' F), \]  
and other higher derivatives as well.

Fig. 16. Conductivity variations in the shaded area of Fig. 14. The curve parameter is the value of $z_1$ (in km). For $z_1 = 468$ km $\sigma(z)$ is a $\delta$-function; for $z_1 = 424$ km $\sigma(z)$ vanishes.

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Eq. (50) indicates that $F$ satisfies a linear third-order differential equation. Since $w(z) = 0$ for $z < z_I$, eq. (15) simplifies to

$$D(z) = \mathbb{R} \sum_{j=1}^2 \lambda_j F_j(z).$$

(52)

$D(z) = 0$ implies that $D''(z) \equiv 0$ in $0 < z < \zeta(z_I)$. Therefore (52) and (50) yield, after division by $4\mu_0\sqrt{\sigma} > 0$, integration, and squaring,

$$\sigma(z) = A \left( \mathbb{R} \sum_{j=1}^2 i\omega \lambda_j F_j(z) \right)^2,$$

(53)

where $A$ is a positive constant of integration. The definition of $\sigma(z)$ by (53) is only implicit, since $F_j(z)$ also depends smoothly on $\sigma(z)$. Let $\mathcal{C}^r(0, \zeta)$ be the set functions, which are $r$ times continuously differentiable at $z \in (0, \zeta)$. If $\sigma(z) \in \mathcal{C}^r(0, \zeta)$, then (49) or (50) implies that $F_j(z) \in \mathcal{C}^{r+2}(0, \zeta)$. From (53) it follows that $\sigma(z) \in \mathcal{C}^{r+2}(0, \zeta)$, and hence that all derivatives of $\sigma(z)$ are continuous at $z \in (0, \zeta)$. Moreover, since $F_j(z)$ is bounded, $\sigma(z) > 0$ for $z \in (0, \zeta)$.

In the present case, the numerical task consists of the determination of the continuous function $\sigma(z)$ and of the two complex and six real discrete parameters $\lambda_1$, $\lambda_2$, $A$, $\zeta(z)$, $\tau(z)$, and $\zeta_{\omega}$. In order to do this, we can use (53) and the two complex and six real equations


c_1[\sigma] = c_1,

c_2[\sigma] = c_2,

$$\mathbb{R} \sum_{j=1}^2 \lambda_j F_j(0) = 0, \quad \mathbb{R} \sum_{j=1}^2 \lambda_j F_j(\zeta) = 0,$$

$$\mathbb{R} \sum_{j=1}^2 \lambda_j F_j(\zeta) = 0, \quad \mathbb{R} \sum_{j=1}^2 \lambda_j F_j(z_I) = 1/\Delta,$$

$$\mathbb{R} \sum_{j=1}^2 \lambda_j F_j(z_2) = 1/\Delta, \quad \mathbb{R} \sum_{j=1}^2 (i\omega \lambda_j) F_j(z_2) = 0.$$

(54)

The last equation is the two-frequency analogue of (24), expressing the insensitivity of the objective function to small displacements of the final perfect conductor at $z = \zeta_{\omega}$. For each evaluation of an equation in the first two rows of (54), the Lagrangian multipliers $\lambda_j$ are computed from the four linear equations in the last two rows and are used to obtain a self-consistent conductivity profile after a few iterative cycles of (53). With these updated values of $\lambda_j$ and $\sigma(z)$, the equations in the first two rows of (54) then serve to determine the remaining six non-linear real parameters.

In the present case the construction of the correct extremal model is of mere theoretical interest, because for $z_1 = 446$ km, $z_2 = 750$ km the incorrect model B1 (Fig. 15) gives $\sigma_{\text{max}} = 0.263028$ S m$^{-1}$, which is only insignificantly smaller than the correct value of $\sigma_{\text{max}} = 0.263072$ S m$^{-1}$ (model B2). The structure of the extremal models in Fig. 14 has been obtained by gradually deforming model I, which is the firm starting point for $z_1 \to 0$, $z_2 \to 0$. The required changes of the extremal models then essentially follow from the changes of the model parameters and of $D(z)$. Typical signals for a change of the model structure are:

- $\tau(z_1) \to 0^+$: omit sheet at $z_1$.
- $D(z_1) \to 0^+$: include sheet at $z_1$.
- $D(z_1) = 0$ and $D'(z_1) \to 0^+$: sheet at $z_1$ tends to move downwards.
- $\zeta \to \zeta_{\omega}$: mobile sheet is fixed at $z_1$.
- $\zeta_{\omega} \to \infty$: mobile sheet merges with terminating perfect conductor.

The notation $D(z_1^+) \to 0^+$ etc. means that $D(z_1^+)$ tends to zero through positive values. In addition, there are analogues to the first four conditions for the surface sheet, which involve replacing $z_1$ by 0. The following two model changes are slightly problematic.

1. The necessity of introducing the perfect conductor for $z_2 > 386$ km cannot be detected when the boundary is approached from above; all necessary conditions can be satisfied without this conductor also for $z_2 > 386$ km. The boundary is detected only by the diving perfect conductor when approaching it from below. This is an example where the necessary conditions (16) are not sufficient. The anomaly is connected with the fact that in both cases $D(z) \equiv 0$ below the mobile sheet.

2. When traversing the boundary from $(z_1, z_2) = (0, 460)$ km to $(200, 660)$ km from the right, the surface sheet vanishes and the sheet at $z_1$ becomes mobile at the same time. This double change is required, because otherwise the model on the left would have had only three free parameters. At this boundary the continuous change of the extremal model is accompanied by a discontinuous change of $D(z)$.

In all thin-sheet cases the determination of the model parameters is simple. Consider as an example the region in Fig. 14 that is the right neighbour of the continuous conductivity region. The model parameters agree with those of that region, except that the continuous conductivity section is replaced by a thin sheet of conductance $\tau$ at $z = \zeta_{\omega}$. Therefore the set of necessary equations is (54) with $F_j(\zeta)$ replaced by $\zeta_{\omega}$. The six real equations in the first two lines form a non-linear system of equations for the determination of the six model parameters, provided that, at each call of one equation in the second line, the two complex Lagrangian multipliers $\lambda_j$ are determined by solving the linear system consisting of the two real equations in the last two lines of (54).

After the detailed discussion of $\sigma_{\text{max}}(z_1, z_2)$, we turn our attention only briefly to $\sigma_{\text{max}}(z_1, z_2)$ (Fig. 17). The minimum conductance model II is a secure starting point for $z_1 \to 0$, $z_2 \to \infty$. The unexpected feature is the discontinuous change of the extremal models along the two barred lines. In a strip along these border lines, both models satisfy the necessary extremal conditions. The shape of the boundary is therefore found only by comparing the actual values of $\sigma(z_1, z_2)$.

4.1.2 General structure of the unconstrained extremal models

The two-frequency example of the previous section, requiring a continuous conductivity layer in $0 < z < z_1$, shows that the unconstrained multifrequency extremal models in general no longer lie in the class of thin-sheet models, which is the only type we encountered for one frequency. Negative side-lobes of $D(z)$ in multifrequency thin-sheet models signal the necessity of introducing a continuous conductor. The discussion of the multifrequency case has to consider separately the structures of the extremal models in $z > z_1$ and $z < z_2$. © 1995 RAS. GJI 123, 683–714
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Figure 17. Structure of the extremal models $\sigma_{\text{min}}(z_1, z_2)$ for a two-frequency problem. A discontinuous model change occurs across the barred lines. The models giving $\sigma_{\text{min}}(z_1, z_2) = 0$ are non-unique.

(a) Structure in $z > z_2$
Assume first that no continuous conductivity variation exists in $z > z_2$. Then the model is terminated either by a perfect conductor at $z = \zeta_\infty$ or by an insulator, and $D(z) \to 0$ and $D'(z) \to 0$ for $z \to \zeta_\infty$ or $z \to \infty$. Between sheets, $D(z)$ varies quadratically. Let the deepest finite-conductance sheet be at $z = \zeta_n$. If it is mobile, then $D(\zeta_n) = 0$ and $D'(\zeta_n) = 0$ imply that $D(z) \equiv 0$ for $z \geq \zeta_n$ and $D'(\zeta_n) = D'(\zeta_n) = 0$. If the adjacent sheet at $z = \zeta_{n-1}$ is also mobile, then $D(\zeta_{n-1}) = 0$ implies that $D(z) \equiv 0$ for $z \geq \zeta_{n-1}$. Continuing these arguments, we infer that $D(z) \equiv 0$ for $z \geq \zeta_1$, where $\zeta_1$ is the position of the first sheet in $z > z_2$. For $\sigma_{\text{min}}(z_1, z_2)$ we have $\zeta_1 = z_2^1$. In $z > \zeta_1$, the necessary extremal condition $D(z) \geq 0$ for $\sigma(z) = 0$ can be realized only in its weak sense.

Now we have to investigate whether the thin-sheet structure can merge at some depth level $\zeta$ (e.g. immediately above a sheet) into a continuous conductivity structure, which also
requires \( D(z) = 0 \). The results of this investigation can be summarized as follows.

1. No continuous conductivity structure can exist in \( z \geq z_2 \).
2. In \( z > z_2 \), at most \( M - 1 \) mobile sheets occur, possibly terminated by a perfectly conducting sheet.
3. A maximum number of \( M - 1 \) mobile sheets (and a terminating perfect conductor) can occur in the extremal models for \( \sigma_{\text{max}}(z_2, z) \) in the limit \( z_2 \rightarrow 0 \).
4. Thin sheets are required for \( \sigma_{\text{max}} \) at \( z = z_2^+ \) and for \( \sigma_{\text{min}} \) at \( z = z_2^- \).

In what follows, Propositions 1 and 2 are proved by showing that the converse assumption leads to a contradiction. In particular we try to establish, on the basis of the converse assumption, \( 2M \) homogeneous linear equations:

\[
\Re e \sum_{j=1}^{M} i(\omega_m)^j \psi_j = 0, \quad n = 0, \ldots, 2M - 1, \quad (55)
\]

for the complex quantities \( \psi_j \), which decouple into two systems of \( M \) equations for \( \Re e \psi_j \) and \( \Im m \psi_j \) with the non-singular Vandermonde determinant \( \det (\omega_m^j)_{j=1}^{M}, n = 0, \ldots, M - 1 \). Hence, (55) admits only the trivial solution \( \psi_j = 0 \), which will then invalidate the converse assumption.

**Proof of Proposition 1**

The proof essentially follows W1. Assume that \( \sigma(z) \) shows a transitional variation \( 0 < \sigma(z) < \infty \) in \( z \in (a, b) \), \( a \geq z_2 \), and let \( \Phi_j(z) = J_j F_j(z) \). Then according to (15) and (16)

\[
\Re e \sum_{j=1}^{M} \Phi_j(z) = 0, \quad \text{for } z \in (a, b). \quad (56)
\]

The continuous section cannot reach to infinity, i.e. \( b < \infty \).
For a proof, assume the converse, differentiate (56) three times with respect to \( z \), divide by \( 4\mu_0 \sqrt{\sigma} \), and integrate from \( z \) to \( \infty \). Using (50) one obtains

\[
\Re e \sum_{j=1}^{M} i(\omega_m)^j \sqrt{(\sigma(z))} \Phi_j(z) - \Re e \sigma(\infty) \Phi_j(\infty)] = 0, \quad z > a. \quad (57)
\]

A comparison with the Bessel function cases behaving as \( \sigma(z) \sim z^r, \quad -\infty < m < \infty \), for \( z \rightarrow \infty \) reveals that it is always the case that

\[
\lim_{z \rightarrow \infty} \sqrt{(\sigma(z))} \Phi_j(z) = 0.
\]

Therefore (57) reduces to

\[
\Re e \sum_{j=1}^{M} i(\omega_m)^j \Phi_j(z) = 0, \quad z > a.
\]

A \((2M-2)\)-fold repetition of the same process then leads to (55) with \( \psi_j = \Phi_j(z) \). Since \( |F_j(z)| \) is a continuously decaying function, \( \Phi_j(z) = 0 \) implies that \( \lambda_j = 0 \), such that the extremal conductivity averages would be independent of the data. Hence, \( b < \infty \).

Now assume that \( z = b < \infty \) is the lower boundary of the continuous conductivity section. In the unconstrained case, only a stack of \( K \) mobile thin sheets can exist in \( z \geq b \), possibly terminated by a perfect conductor. First let \( K > 0 \). Then the insensitivity of the objective function \( Q \) to a small change of conductance \( \tau_k \) and position \( \zeta_k \) of sheet \( k \) requires that \( D(\zeta_k) = 0, \quad D^\prime(\zeta_k) = 0 \), which yields, using (A9),

\[
\Re e \sum_{j=1}^{M} R_n(\omega_m) \Phi_j(\zeta_k) = 0, \quad n = 0, \ldots, 2K - 1, \quad (58)
\]

where \( R_n(x) \) is a polynomial in \( x \) of exact degree \( n \), i.e. the leading coefficient does not vanish. If \( K < M \), additional \( (2M - K) \) equations are generated by considering the field at \( z = b \). According to the discussion after eq. (53), \( \sigma(z) \) has derivatives of any order and \( \sigma(b^-) > 0 \). Assume that the shallowest sheet lies at \( z > b \). Then the situation of Appendix A applies with \( z_0 = b \) and \( d_j > 0 \). At \( z = b \) the fields \( F \) and \( F' \) are continuous. Eqs (A10), (A12) and (51) yield

\[
F(b) = R_{2K}(\omega_m)F(\zeta_k), \quad F'(b^-) = R_{2K+1}(\omega_m)F'(\zeta_k),
\]

and more generally

\[
F^{(2m)}(b^-) = R_{2K+m}(\omega_m)F^{(2m)}(\zeta_k). \quad (59)
\]

Since all derivatives of \( D(z) \) vanish at \( z = b^- \), the system (58) can be extended to \( n = 2M - 1 \). Because \( R_n \) is of exact degree \( n \), it is possible to diagonalize the extended system (58) by subtracting a suitable multiple of equation \( n \) from equation \( k > n, n = 0, \ldots, M - 2 \), yielding

\[
\Re e \sum_{j=1}^{M} i(\omega_m)^j \Phi_j(\zeta_k) = 0, \quad n = 0, \ldots, 2M - 1. \quad (58)
\]

Hence, \( \Phi_j(\zeta_k) = 0 \) and \( \lambda_j = 0 \). Consequently, under the above assumptions no continuous conductivity exists in \( z > b \).

Only slight modifications are necessary to reach the same conclusion for \( K = 0 \) or \( d_j = 0 \). For instance, if in the case \( K = 0 \) the infinitely conducting sheet is at \( z = b \), then \( F(b) = 0 \), \( F'(b) = 0 \), but \( F''(b^-) = 2i\mu_0 [f'(b)]^2 \neq 0 \). Hence

\[
F^{(2m+2)}(b^-) = R_{2K}(\omega_m)F^{(2m+2)}(\zeta_k). \quad (59)
\]

With \( \psi_j = \Phi_j'(b^-) \) it then follows from (55) that \( \psi_j'(b^-) = 0 \), and therefore that \( \lambda_j = 0 \). On the other hand, assuming that the continuous section starts immediately above a sheet, i.e. \( d_j = 0 \), then, in view of the remark at the end of Appendix A, eq. (59) is replaced by

\[
F^{(2m+2)}(b^-) = R_{2K+1}(\omega_m)F^{(2m+2)}(\zeta_k). \quad (59)
\]

**Proof of Proposition 2**

If there were more than \( M - 1 \) mobile sheets below \( z_2 \), then the homogeneous equations resulting from (58) would, via (55) with \( \psi_j = \Phi_j(\zeta_k) \), immediately lead to \( \lambda_j = 0 \).

**Proof of Proposition 3**

An exact \( M \)-frequency set is called degenerate if it results from a thin-sheet structure with less than \( 2M \) free parameters. In this case only one conductivity model exists. It is shown in W1 and by Yee & Paulson (1988) that exact, non-degenerate \( M \)-frequency data admit the representation

\[
c(\omega_j) = \sum_{m=1}^{M} a_m \lambda_m + i\omega_m \quad j = 1, \ldots, M,
\]

\( a_m > 0, \lambda_m > 0 \), which can be interpreted in terms of a surface sheet, \( M - 1 \) mobile sheets, and a terminating perfect conductor. The surface sheet of this model has the greatest conductance of all surface sheet models fitting the data. Therefore this model is attained by \( \sigma_{\text{max}} \) for \( z \rightarrow 0 \).
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(c) Structure in \( z < z_2 \)

Weidelt (1985 i.e. W1) considered extremal conductance models \( (z_1 = 0) \) in the unconstrained case and concluded that no continuous conductivity variations could exist in \( z > z_1 \). However, a re-examination has shown that his conclusions are valid only for \( z > z_2 \) (see above), whereas in addition to in the range \( 0 < z < z_1 \)—continuous conductivity variations can no longer be ruled out in \( z_1 < z < z_2 \) either. The error in W1 occurred in his eq. (A10), where the fact that \( D(z) \) is discontinuous across the sheet at \( z = z_2 \) was overlooked. In what follows, two-frequency data sets are generated which require a continuous conductivity variation also in \( z_1 < z < z_2 \). Two simple examples are shown in Fig. 18.

In the example for \( \sigma_{\text{max}}(z_1, z_2) \) (A) we arbitrarily assign values to \( \omega_1, \omega_2, \zeta_\infty, z_1, z_2, \tau(z_2) \) and \( \sigma(z_2) \), then determine \( \sigma(z) \) in \( z_1 < z < z_2 \) by satisfying all the necessary extremal conditions, and finally calculate \( c_1 \) and \( c_2 \). Once again let \( \Phi_j(z) = \lambda_j F_j(z) \). The necessary extremal condition \( D(z) = 0 \) for \( z \in (z_1, z_2) \) requires in particular that \( D^{(n)}(z_2) = 0 \) for \( n \geq 0 \). The first four conditions are

\[
\begin{align*}
\Re \sum_{j=1}^2 \Phi_j(z_2)/i\omega_j &= 0, \quad \Re \sum_{j=1}^2 \Phi_j(z_2) = 1/D, \\
\Re \sum_{j=1}^2 \Phi'_j(z_1) &= 0, \quad \Re \sum_{j=1}^2 \Phi''_j(z_1) = 0, \quad (60)
\end{align*}
\]

Proof of Proposition 4

This has already been proved in Section 3.3 of W1.

Figure 18. Two examples of extremal models with a continuous conductivity section in \( z_1 < z < z_2 \). A: \( \sigma_{\text{max}}(z_1, z_2) \), B: \( \sigma_{\text{max}}(z_1, z_2) \). The response B is dominated by the thin sheet at \( z_2 \).

with

\[
\begin{align*}
\Phi'(z_1^+)/\Phi(z_2) &= -2/c(z_1^+), \\
\Phi'(z_1^-)/\Phi(z_2) &= 2i\omega_0 \sigma(z_1^-) + 2/c(z_1^-), \quad \text{ (61)}
\end{align*}
\]

and \( 1/c(z_1^-) = i\omega_0 \tau(z_1^-) + 1/(\zeta_\infty - z_2^-) \). The four linearly independent equations can be solved for \( \Phi_j(z_2) \). One possibility for the determination of \( \sigma(z) \) is then a Taylor expansion at \( z = z_2 \), where—continuing the sequence (49) to (51)—the derivatives \( \sigma^{(n)}(z_2^-) \) of the infinitely differentiable function \( \sigma(z) \) are obtained from \( D^{(n+2)}(z_2^-) = 0 \). For instance, \( D^{(2)}(z_2^-) = 0 \) yields

\[
\sigma'(z_2^-) = -2\sigma(z_2^-) \Re \sum_{j=1}^2 i\omega_j \Phi_j(z_2^-) \left/ \Re \sum_{j=1}^2 i\omega_j \Phi_j(z_2^-) \right.,
\]

which after insertion of (61) and (60) simplifies to

\[
\sigma'(z_2^-) = -4\sigma(z_2^-)/c(z_2^-). \quad \text{ (62)}
\]

The radius of convergence of the Taylor series might be very small. Therefore it is better to obtain \( \sigma(z) \) implicitly by (53), which in the present context reads

\[
\sigma(z) = \sigma(z_2^-) \left\{ \Re \sum_{j=1}^2 i\omega_j \Phi_j(z_2^-) \left/ \Re \sum_{j=1}^2 i\omega_j \Phi_j(z_2^-) \right. \right\}^2. \quad \text{ (63)}
\]

After a few iterative cycles starting with \( \sigma(z) = \sigma(z_2^-) \), eq. (63) quickly converges to a self-consistent conductivity profile in \( z_1 < z < z_2 \), which can then be used to compute \( c_f \). The approximate thin-sheet model based on this data shows thin sheets at \( z_1 = 100 \) km and at \( \zeta = 124 \) km with negative side-lobes of \( D(z) \) in \( z_1 < z < \zeta \) and \( \zeta < z < z_2 \), whereas in the true model \( \sigma(z) \) decreases continuously from \( \sigma(z_1^-) = 605 \text{ mS m}^{-1} \) to \( \sigma(z_2^-) = 10 \text{ mS m}^{-1} \). Again the approximate maximum
245.097 mS m\(^{-1}\) cannot be distinguished from the true maximum \(\sigma_{\text{max}}(Z_1, Z_2) = 245.115 \text{ mS m}^{-1}\).

The continuous conductivity model for \(\sigma_{\text{min}}(Z_1, Z_2)\) (Fig. 18, panel B) is constructed similarly. Assigning arbitrary values to \(\omega_1, \omega_2, \gamma_1, \gamma_2, Z_1, Z_2, \tau(z_2)\) and \(\sigma(\gamma_2)\), the first four conditions are

\[
\begin{align*}
\mathcal{K} & \sum_{j=1}^{2} \Phi_j(z_2) = 0, \quad \mathcal{K} \sum_{j=1}^{2} \Phi_j(\gamma_2) = -1/\Delta, \\
\mathcal{K} & \sum_{j=1}^{2} \Phi_j(z_2) = 0, \quad \mathcal{K} \sum_{j=1}^{2} \Phi_j(\gamma_2) = 0,
\end{align*}
\]

augmented by an equation analogous to (61) with \(z_3\) replaced by \(\gamma_z\). Moreover, \(\Phi(z_2)/\Phi(\gamma_2) = [c(z_2)/c(\gamma_2)]^2\), with \(c(\gamma_2) = z_2 - \gamma_z + c(z_2)\) and \(1/c(\gamma_2) = \text{im} \sigma(\gamma_2)\). The relation corresponding to (61) is \(\sigma(\gamma_2) = 4\sigma(\gamma_1)/(z_2 - \gamma_z)\). In the present example, \(\sigma(z)\) increases continuously with depth. Again there is a negligible difference between the approximate and the true minimum (16.20 and 16.00 mS m\(^{-1}\), respectively).

### 4.1.3 Number of thin sheets

Of particular interest is the number of sheets, \(N\), that can be expected for \(M\) frequencies. A general answer to this question turns out to be difficult. An inspection of Figs 1, 4, 14 and 17 (excluding the continuous section of Fig. 14) shows that for \(M = 1, 2\) this number varies between \(M + 3\) and \(M + 3\).

If we are fitting \(M\) exact responses, the extremal models must be represented by at least \(2M\) free parameters. Since the omnipresent sheet at \(Z_3\) is described by only one parameter, in general at least \(N = M + 1\) sheets are required. This number is reduced to \(N = M\) only in the special case when \(z_3\) coincides with a level of model II (for example eq. 48) in which the responses are represented by \(M\) thin sheets (Weidelt 1985). In Fig. 14 this situation occurs for \(z_3 = \gamma_z = 214\) km. If no continuous conductor occurs, the upper limit appears to be \(N = M + 3\): for \(\sigma_{\text{max}}(z_1, z_2), M = 1\) mobile sheets and four one-parameter sheets at \(z = 0, Z_1, Z_2, \) and \(\gamma_z\), and, for \(\sigma_{\text{min}}(z_1, z_2), M = 1\) mobile sheets and up to three one-parameter sheets. However, no general proof could be obtained for this assertion. Therefore, the condition that a continuous conductivity section can be understood as an infinite sequence of thin sheets, the continuous section of B2 in Fig. 15 can be approximated by an arbitrary number of mobile thin sheets, connected, however, by small negative side lobes. Therefore, in the general case no finite upper bound of \(N\) exists.

In the physically realistic case of noisy data considered in the next section, where the interpretation is based on a global misfit constraint, the extremal models in general will show much less structure and even the minimum of \(M + 1\) sheets may not be reached.

### 4.1.4 Noisy data

So far only exact multifrequency data have been considered. However, the analysis of noisy data under the \(\chi^2\)-constraint (4) requires only a few changes. The control function is now

\[
D(z) := w(z) + 2M \sum_{j=1}^{M} \hat{\lambda}_j F_j(z), \quad (64)
\]

with \(\hat{\lambda}_j\) defined in (14). Assuming the normal case that the \(\chi^2\)-bound is active, i.e. (4) is satisfied as an equality, the data induce only one (positive) Lagrangian multiplier \(\Lambda\) rather than \(M\) complex multipliers \(\lambda_j\). For \(M\) frequencies the structure of the extremal models for noisy data will in general be simpler than that for exact data, since the resulting model will tend to approximate an exact-data case for fewer than \(M\) frequencies, which yields a greater \(\sigma_{\text{max}}\) and a smaller \(\sigma_{\text{min}}\) than the exact \(M\)-frequency case. If a solution has been found for one pair \((z_1, z_2)\), a solution for other pairs will be reached by deforming the original solution and monitoring the change of \(D(z)\).

The numerical procedure for noisy data is described by an example. Assume that the extremal model for \(\sigma_{\text{max}}(z_1, z_2)\) requires three finite conductance sheets at \(z_1, z_2,\) and \(\gamma_z\). Then the set of five necessary conditions for the determination of \(\tau(z_1), \tau(z_2), \tau, \zeta\) and \(\Lambda\) is

\[
\sum_{j=1}^{M} |c_j - c_j[\sigma]|^2 / \delta_j^2 = B, 
\]

\[
\mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(z_1) = 1/\Delta, \quad \mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(z_2) = 1/\Delta, \quad (65)
\]

\[
\mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(\tau) = 0, \quad \mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(\gamma_z) = 0.
\]

Using (14), let

\[
\hat{\lambda}_j = 2\Delta \hat{\lambda}_j, \quad \hat{\lambda}_j = (c_j - c_j[\sigma])^*/\delta_j^2. \quad (66)
\]

Then \(\Lambda\) can be eliminated from all except one equation by rewriting the last four equations of (65) as

\[
\mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j [F_j(z_2) - F_j(z_1)] = 0,
\]

\[
\mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(\zeta) = 0, \quad \mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(\tau) = 0, \quad \mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(\gamma_z) = 0, \quad (67)
\]

\[
\Lambda = 1 \left\{2M \mathcal{K} \sum_{j=1}^{M} \hat{\lambda}_j F_j(z_2) \right\} > 0.
\]

For given model parameters \(\tau(z_1), \tau(z_2), \tau, \zeta\) and \(\Lambda\), the coefficients \(\hat{\lambda}_j\) defined by (66) can be calculated immediately. Therefore the first equation of (65) along with the first three equations of (67) constitute a system of four non-linear equations for the determination of the model parameters. \(\Lambda\) then follows from the last equation of (67), and is used via (66) and (64) to obtain \(D(z)\), which then may suggest further action.

### 4.2 Constrained conductivity models

The constrained multifrequency case will be briefly discussed via an example only. Fig. 19 shows the structure of the models of \(\sigma_{\text{max}}(z_1, z_2)\) for the two-frequency data (47) under the constraints \(\sigma_- = 0.01 \text{ S m}^{-1}\), \(\sigma_+ = 1 \text{ S m}^{-1}\). A comparison with the unconstrained case, Fig. 14, reveals a more regular structure of the constrained models: the two-storey structure for small \(z_2\) (induced by the two frequencies) becomes more obvious, no continuous conductor is required, and the conducting substructure cannot disappear (as in Fig. 14 for 214 km < \(z_2 < 386\) km) since \(\sigma_+ > 0\) always requires a layered substructure. For given \(z_2\), the shaded region at the diagonal shows the thickness of the conductor overlying \(z = z_2\), which coalesces at \(z_2 \approx 950\) km with the substructure (compared with 750 km for the longer period only; see Fig. 9). At the shallower level of \(z_2 \approx 500\) km, two layers in \(z > z_2\) have already merged. The terminating dots denote a quasi-periodic sequence of layers, which soon approaches the periodic \(\lambda/4\)-sequence of the longer period.

A more quantitative picture of the conductivity distribution...
Local electrical conductivity averages

$\bar{\sigma}_{\text{max}}(z_1, z_2)$

$\sigma_0 = 0.01 \text{ S/m}$

$\sigma_6 = 1 \text{ S/m}$

$T_1 = 24 \text{ h, } c_1 = (550 - 275i) \text{ km}$

$T_2 = 6 \text{ h, } c_2 = (350 - 220i) \text{ km}$

$\sigma_{\text{max}}(z_1, z_2)$

$\sigma = 0.01 \text{ S/m}$

$\sigma_6 = 1 \text{ S/m}$

Figure 19. Structure of the constrained two-frequency extremal models $\bar{\sigma}_{\text{max}}(z_1, z_2)$. The structure is more regular than in the unconstrained case (Fig. 14). Owing to the well-separated two frequencies, the model structure in $0 < z_2 < 500 \text{ km}$ is repeated below (with the mobile conducting layer now between $z_1$ and $z_2$).

for various values of $z_2$ is drawn in Fig. 20. For a fixed $z_1$ of 100 km and a varying $z_2$ (vertical scale), the layering in the $z$-direction is shown along the horizontal scale, where the conducting layers are shaded. It demonstrates nicely how, with increasing $z_2$, the conductor below $z = z_2$ merges with the substructure and at the same time a new conductor evolves at $z = z_1$, becomes detached, and moves downwards. This latter descent then activates a surface layer to keep the centre of gravity of induced currents at $c = 350 \text{ km}$ for the shorter period. At $z_2 \approx 350 \text{ km}$ the conducting substructure dives down to a depth of about 2300 km and reappears for increasing $z_2$. This figure also illuminates the topological structure of the conducting regions in the neighbourhood of the two levels of coalescence at $z_2 \approx 500 \text{ km}$ and $z_2 \approx 950 \text{ km}$.

Finally, isolines of $\sigma_{\text{max}}(z_1, z_2)$ are displayed in Fig. 21. They show the conductivity between the surface and $z_2 = 175 \text{ km}$ with $\sigma_{\text{max}} < 25 \text{ mS m}^{-1}$ as the best-constrained feature (compared with $40 \text{ mS m}^{-1}$ in the absence of a priori conductivity bounds).

The structure of the external models in Fig. 19 has been obtained by systematically deforming the solution for small $z_2$ and monitoring $D(z)$. Because of the more regular structure of the constrained extremal models no problems were encountered. The following signals typically indicate that a change of model is necessary.

(1a) $\sigma(z_1) = \sigma_+, D(z_1) \rightarrow 0^-$; conducting layer underlying $z_1$ vanishes $[\sigma(z_1 + \varepsilon) = \sigma_-]$ or tends to move downwards $[\sigma(z_1 + \varepsilon) = \sigma_+]$.

(1b) $\sigma(z_1) = \sigma_-, D(z_1) \rightarrow 0^+$; conducting layer emerges at $z_1 [\sigma(z_1 + \varepsilon) = \sigma_-]$ or reaches $z_1$ from below $[\sigma(z_1 + \varepsilon) = \sigma_+]$.

(2a) $\sigma(z) = \sigma_-, D'(z) = 0, D(z) \rightarrow 0^-$: two conducting layers coalesce at $z$.

(2b) $\sigma(z) = \sigma_+, D'(z) = 0, D(z) \rightarrow 0^+$: conducting layer splits at $z$.

(3a) $\sigma(z_2) = \sigma_-, D(z_2) \rightarrow 0^+$; conducting layer overlying $z_2$ merges with the substructure.

(3b) $\sigma(z_2) = \sigma_+, D(z_2) \rightarrow 0^-$: conductor splits at $z_2$.

Here $\rightarrow 0^-(\rightarrow 0^+)$ means that 0 is reached through positive (negative) values; $\varepsilon$ is a small positive length which does not vanish in the limit under consideration. The subcases (a) and
Figure 20. Quantitative structure of the extremal models $\sigma_{\text{max}}(z_1, z_2)$ for $z_1 = 100$ km and varying $z_2$ (ordinate). The horizontal axis shows the sequence of poorly and well conducting layers (shaded). The depth levels $\zeta = \eta_i$ mark the 'centres of gravity' of the induced currents.

(b) require inverse actions. Case 1(a,b) also holds for the surface sheet. Although no continuous conductors were required in the simple example of Fig. 19, they will occur for other data sets and/or other a priori bounds. In particular, they can no longer be ruled out also in $z > z_2$. This can be anticipated from the fact that in the example, for $z_2 \approx 500$ km, two conducting layers coalesce at $\zeta \approx 800$ km (see Fig. 20), which requires $D(\zeta) = 0$, $D'(\zeta) = 0$ [see also case 2(a,b) above]. These are also the initial conditions for a continuous conductivity section overlying the level $\zeta$. Therefore a different data set could require a continuous conductor also in $z > z_2$.

5 CONCLUSIONS

The scope of this paper is the investigation of how much incomplete data constrain local averages of the electrical conductivity in the simple 1-D magnetotelluric inverse problem. This task is cast into variational form and exact bounds on the average conductivity are derived in a fully non-linear treatment, at the expense of handling extended systems of non-linear algebraic equations. Therefore the exact method appears to be suitable only for data sets of modest size, whereas an approximate method (Section 3.2.4) can be applied to data sets of any size.

Layered conductors whose response is compatible with a given data set are constructed; they accommodate in a prescribed depth range $z_1 \leq z \leq z_2$ either as much or as little conducting material as possible. The conductivity may be constrained by a priori bounds $\sigma_+ \leq \sigma_{\text{c}}(z) \leq \sigma_-$. In particular for more than one frequency, the topological structure of the extremal model is very sensitive to the depth interval $[z_1, z_2]$ (see Figs 14, 17 and 19). The control function $D(z)$ associated with the variational problem signals when changes in the model structure are necessary for varying pairs $(z_1, z_2)$. In the unconstrained case, occasionally two competing models were found in some strip of the $(z_1, z_2)$-plane which both satisfied the necessary extremal conditions controlled by $D(z)$. Only by comparing the actual values of $\sigma(z_1, z_2)$ could the true extremum be identified. Additional work is certainly required in this field. In most cases considered, the extremal models consist of a sequence of thin sheets (or in the constrained case of a sequence of layers with alternating conductivities $\sigma_-$ and $\sigma_+$). For more than one frequency, the exact extremal models may require sections with a continuous conductivity variation. However, due to the insensitivity of magnetotelluric responses to vertical conductivity gradients in the examples studied, the exact extremes could not be distinguished from the approximate bounds, where the conductivity switches between its extremes.

The isoline maps of the extremal average conductivities $\sigma_{\text{max}}(z_1, z_2)$ and $\sigma_{\text{min}}(z_1, z_2)$ for one- and two-frequency data sets simulating $Sq$-responses show, with small values for $\sigma_{\text{max}}$ and large values for $\sigma_{\text{min}}$, in which portions of the conductor the average conductivity is sensibly constrained by the data. In particular, the tightest upper bounds can be obtained for the conductivity average between the surface and a depth corresponding to one-half of the smallest real part of the $c$-responses used. In the $Sq$-data case, the less constrained lower bound is tightest for a conductivity rise in this depth range. Although the data basis considered is very small, even two frequencies may provide useful constraints for the upper bound of the average conductivity, in particular if in addition reasonable a priori constraints are incorporated (Fig. 21).

The prime interest in extremal models lies not in their actual...
structure, but in the bounds they provide. However, the
complexity of the problem has required an exploratory study
focusing on the non-linear features of the model evolution,
which even for simple data sets has revealed an unexpected
dynamical behaviour.

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APPENDIX A: COLLECTION OF FORMULAE FOR STACKS OF THIN SHEETS

Consider, as in Section 3.2.1, a conductivity model consisting below a level $z = \zeta_0$ of a stack of $K$ thin sheets of finite conductance $\tau_k$ at depth $z = \zeta_k - 1$.

$$\sigma(z) = \sum_{k=1}^{K} \tau_k \delta(z - \zeta_k), \quad (A1)$$

terminated possibly by an additional perfectly conducting sheet at $\zeta_{K+1}$.

The inter-sheet separations are $d_k = \zeta_k - \zeta_{k-1}$, $k = 1, \ldots, K + 1$. The integration of (1) across $\zeta_k$ using (A1) yields the jump relation (18):

$$f'(\zeta_k) - f'(\zeta_{k-1}) = \tau_k \sigma \tau_k . \quad (A2)$$

Let $f_k = f(\zeta_k)$. Since according to (1) and (A1) $f(z)$ varies linearly between sheets, (A2) reads explicitly

$$\frac{f_{k+1} - f_k}{d_k} - \frac{f_k - f_{k-1}}{d_k} = \tau_k \sigma \tau_k . \quad (A3)$$

Let

$$c_k^+ = -\frac{f_{k+1} - f_k}{d_k}, \quad c_k^- = -\frac{f_k - f_{k-1}}{d_k} \quad (A4)$$

be the values of the response function $c$ below and above sheet $k$. Then (A3) and (A4) yield, as cross-sheet and inter-sheet variations of $c$,

$$\frac{1}{c_k^+} = 1/c_k^- + \frac{\tau_k \sigma \tau_k}{d_k}, \quad (A5)$$

$$c_k^* = c_k^+ + c_k^- \quad (A6)$$

leading to the recursion

$$\frac{1}{c_k^*} = \frac{\tau_k \sigma \tau_k}{d_k}. \quad (A7)$$

starting with $c_1^- = 0$. If the perfectly conducting sheet is missing, $d_{K+1} = \infty$.

The multifrequency case in Section 4.1.2 requires some knowledge of the frequency dependence of $f(z)$ and the Fréchet derivative $F(z)$. From (A3) it follows that

$$f_{k+1} - f_k = \frac{d_k}{d_k} - \frac{f_k - f_{k-1}}{d_k} . \quad (A8)$$

According to whether the perfectly conducting sheet at $z = \zeta_{k+1}$ is present or absent, $f_{k+1} = 0$ or $f_{k+1} = f_k$. In both cases, (A8) implies that $f_{k+1}/f_k$ is a first-order polynomial in $\omega \tau_k$.

Recursive application of (A8) then yields

$$f_{K-m} = \frac{R_m(\omega \tau_k) f_k}{m = 0, \ldots, K-1},$$

where $R_m(x)$ denotes a polynomial of exact degree $m$ in $x$. The Fréchet derivative $F(z) = -i\omega \tau_k f^2 z$ and its averaged slope are then given by

$$F(\zeta_{K-m}) = R_{2m}(\omega \tau_k) f(\zeta_k), \quad F'(\zeta_{K-m}) = R_{2m+1}(\omega \tau_k) F(\zeta_k), \quad (A9)$$

since in

$$\left(f^2 \right)_k = f_k \left[ \frac{f_{k+1} - f_k}{d_k} + \frac{f_k - f_{k-1}}{d_k} \right]$$

the term $f_k f_{k-1}$ for $k = K - m$ equals $R_{2m+1} f_k^2$.

Section 4.1.2 also requires the frequency structure of the fields at the top of the stack at $z = \zeta_0$. First assume $d_1 = \zeta_1 - \zeta_0 > 0$. Then

$$F(\zeta_0) = -i\omega \tau_0 f_1^2 = R_{2K}(\omega \tau_k) F(\zeta_0), \quad (A10)$$

$$F'(\zeta_0) = -2i\omega \tau_0 f_0 (f_1 - f_0)/d_1 = R_{2K}(\omega \tau_k) F(\zeta_0), \quad (A11)$$

where the particular structure of $R_{2K}(\omega \tau_k)$ changes from case to case. For $d_1 = 0$ we replace $(f_1 - f_0)/d_1$ by (A3) for $k = 1$. Then (A10) and (A11) are, respectively, polynomials of degree $2K - 2$ and $2K - 1$ only.

APPENDIX B: THE PARAMETERS OF THE ONE-FREQUENCY EXTREMAL MODELS

B.1 Extremal models for $\sigma_{\text{max}}$

The following abbreviations, which were introduced at an earlier stage, are used:

$$z_{1Q} = \frac{h|c|^2}{g + h}, \quad z_{2Q} = \frac{|c|^2}{g + h},$$

$$z_{2M} = \frac{|c|^2}{g}, \quad \Delta = z_2 - z_1,$$

where $(z_{1Q}, z_{2Q})$ are the coordinates of the quadruple point $Q$, $z_{2M}$ is the depth of the shallowest perfect conductor, and $\Delta$ is the averaging range.

The boundaries $z_k = \Phi(z_k)$ between the four regions shown in Fig. 1 can be determined according to the principles stated in Section 3.2.2:

(a) Boundary A–B

$$z_k = \Phi_{AB}(z_k) = g - x,$$

where $x$ is the positive solution of the cubic equation

$$x^3 + (g - z_1)x^2 + 3h x - (g - z_1)h^2 = 0.$$
(b) Boundary B–C
\[ z_2 = \Phi_{BC}(z_1) = z_1 + \frac{x_+ + x_- - \sqrt{a_+ (x_+ - x_-)}}{2(g - z_1)|c|^2}, \]
where
\[ a_2 = [g(g - z_1) + h(h + z_1)]^2. \]

\( \therefore \) Boundary C–D
\[ z_2 = \Phi_{CD}(z_1) = z_{2M} - z_1. \]

(d) Boundary D–A
\[ z_2 = \Phi_{DA}(z_1) = z_{2Q}. \]

Then the parameters of the extremum models are as follows:

Region A \( \{z_1 < z_{1Q} \text{ and } z_2 < \Phi_{AB}(z_1) \} \) or \( \{z_1 > z_{1Q} \text{ and } z_2 < z_{2Q} \} \)

\[ \tau(0) = \frac{1}{\omega \mu_0} \left[ \frac{h}{|c|} - \left( \frac{1}{z_1} \right) \left( \sqrt{1 + x^2} - x \right) \right], \]
\[ \tau(z_1) = -\frac{1}{\omega \mu_0} \left( \frac{1}{z_2} \right) \left( \sqrt{1 + x^2} - x \right) + x - \sqrt{1 + x^2}, \]
\[ \tau(z_2) = -\frac{x}{\omega \mu_0}. \]

\[ \zeta_\infty = z_2 + \Delta(1 + \sqrt{1 + x^2})/x^2, \]
\[ \theta_{\max}(z_1, z_2) = \frac{\tau(z_1) + \tau(z_2)}{\Delta}. \]

The formally identical expressions in regions B and C for \( \tau(z_2) \) and \( \zeta_\infty \) are a consequence of the validity of (B1) in both regions.

Region D \( \{z_1 < z_{1Q} \text{ and } \Phi_{CD}(z_1) < z_2 < z_{2M} \} \) or \( \{z_1 > z_{1Q} \text{ and } z_{2Q} < z_2 < z_{2M} \} \)

The parameters are
\[ \tau(0) = \frac{1}{\omega \mu_0} \left( \frac{1}{z_{2Q}} \right), \]
\[ \tau(z_2) = \frac{1}{\omega \mu_0} \left( 1 - z_2/z_{2M} \right), \]
\[ \zeta_\infty = -\frac{z_2 z_{2M}}{2z_2 - z_{2M}}, \]
\[ \theta_{\max}(z_1, z_2) = \tau(z_2)/\Delta. \]

If data errors are taken into account, \( \theta_{\max} \) varies in the limits
\[ \frac{|c|^2 - s^2}{|c|^2 - s^2 - z_2(g - s)} \leq \frac{|c|^2 - s^2}{|c|^2 - s^2 - z_2(g + s)}. \]

This corresponds to (36) for Region A.

B.2 Extremal models for \( \theta_{\min} \)

The boundaries \( z_2 = \varphi(z_1) \) between the four regions shown in Fig. 4 can be determined according to explanations given in Section 3.2.3.

(a) Boundary A–B
\[ z_1 = g. \]

(b) Boundary B–C
\[ z_2 = \Phi_{BC}(z_1) = g + \frac{h^2}{g - z_1}. \]

(c) Boundary C–D
Let \( y = [g - z_1 + \sqrt{(g - z_1)^2 - h^2}] / h \) with \( y \to 1^+ \) for \( z_1 \to g - h \).

Then
\[ z_2 = \Phi_{CD}(z_1) = g + \frac{8y^3 h}{(y^2 - 1)(3y^2 + 1)}. \]

Region A \( \{z_1 > g\} \) and Region B \( \{z_1 < g \text{ and } z_2 < \Phi_{BC}(z_1)\} \)
\[ \theta_{\min}(z_1, z_2) = 0. \]

The mobile intermediate sheet has conductance \( \tau \) and lies at
Then the parameters of the extremum models yielding 
\( \sigma_{\min}(z_1, z_2) > 0 \) are as follows.

**Region C** \( \{ \max(0, g - h) < z_1 < g \text{ and } z_2 > \varphi_{bc}(z_1) \} \) or \( \{ 0 < z_1 < g - h \text{ and } \varphi_{bc}(z_1) < z_2 < \varphi_{cd}(z_1) \} \)

Let 
\[
y = (g - z_1) \Delta / |c - z_1|^2,
\]
\[
x = (2y - 1 + \sqrt{y^2 - y + 1}) / (3y).
\]

Then
\[
\tau(z_1) = \frac{1}{\omega \mu_0 \Delta} \left( \frac{h \Delta}{|c - z_1|^2} - \frac{\sqrt{3x^2 - 2x}}{(1 - x)(3x - 1)} \right),
\]
\[
\tau = \frac{1}{\omega \mu_0 \Delta} \frac{\sqrt{3x^2 - 2x}}{x(1 - x)},
\]
\[
\zeta = z_2 - x \Delta,
\]
\[
\tau(z_2) = \frac{1}{\omega \mu_0 \Delta} \sqrt{3x^2 - 2x},
\]
\[
\sigma_{\min}(z_1, z_2) = \tau / \Delta.
\]

The compatibility condition resulting from
\[
\mathcal{A}_{\text{BC}}[\int_{\gamma}(\zeta) / F(z_2)] = 0
\]
is
\[
\tau + 2\tau(z_2) = [\omega \mu_0 \tau(z_2)(z_2 - \zeta)]^2 \tau.
\]

**Region D** \( \{ 0 < z_1 < g - h \text{ and } z_2 > \varphi_{cd}(z_1) \} \)

This region is reached only for \( g > h \). Let 
\[
y = |c - z_2|^2 / h^2, \quad 1 < y < \infty \text{ and let } x \text{ be the positive solution of}\]
\[
1 + yx^2 = 2\sqrt{1 + 2x + 2x^2}.
\]

The existence of this solution is easily verified by a graphical display of both sides. Moreover, \( 1 / \sqrt{y} < x < (1 + \sqrt{1 + 3y}) / y \).

Then
\[
\tau = \frac{x(3 + 2x - yx^2)}{\omega \mu_0 h(yx^2 - 1)},
\]
\[
\zeta = z_2 - \frac{2(z_2 - g)}{3 + 2x - yx^2},
\]
\[
\tau(z_2) = \frac{x}{\omega \mu_0 h},
\]
\[
\sigma_{\min}(z_1, z_2) = \tau / \Delta.
\]

(B2) also holds in Region D.