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Letter to the Editor

On the Difference Between Polarisation and Coherence

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1. Introduction

In the analysis of plane waves – for example in optics or magnetic studies – it is of primary interest to determine the polarisation parameters of the signal from its measured components. Several methods have been presented to deal with this problem in both two- and three-dimensions (Fowler et al., 1967; McPherron et al., 1972; Means, 1972; Samson, 1973; Kodera et al., 1977) but in this letter the remarks are restricted to the two-dimensional case – they are equally valid however in three-dimensions. It should be noted that many workers have applied the techniques of these authors to analyse their own data (for example Paulson, 1968; Rankin and Reddy, 1972; Arthur et al., 1976; Jones, 1977; Kodera et al., 1977; Samson, 1977) and have shown the superiority of these forms of frequency domain analyses to the time domain hodogram.

However, it has occurred to the author that there is not a widespread appreciation of the (somewhat) subtle difference between the definitions of polarisation and coherence. It is the purpose of this letter to show that the distinction between polarised and unpolarised parts is not necessarily consistent with that between coherent and incoherent parts.

2. Basic Theory

The basic theory of polarisation analysis is well treated by Born and Wolf (1964) and is repeated in Fowler et al. (1967). For consistency this paper will follow the notation of Fowler et al. (1967) as far as possible. Any wave field can be characterised by examining the coherence (or cross-spectral) matrix of that field given by

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$$J(\omega)=\begin{bmatrix} J_{xx}(\omega) & J_{xy}(\omega) \\ J_{yx}(\omega) & J_{yy}(\omega) \end{bmatrix} \tag{1}$$

where $J_{ab}(\omega)=\frac{1}{T}\langle A^*(\omega)\,B(\omega)\rangle$

and the x and y subscripts refer to orthogonal components of the signal. From this matrix, the following parameters can be determined

det $[J]$ = determinant of J
Tr $[J]$ = trace (or intensity) of J
$\tan\beta$ = ratio of minor to major axis of the polarisation ellipse,
$\beta>0$ implies right-handed polarisation
$\beta=0$ implies linear polarisation
$\beta<0$ implies left-handed polarisation
θ = direction of the major axis of the ellipse (clockwise round from the x-axis)
γ_{xy}^2 = coherence between the two components
R = ratio of polarised power to total power.

All real signals consist (or can be resolved to consist) of three parts, a completely polarised signal, a completely unpolarised signal and a random noise contribution. These three will be considered separately.

(a) *Completely Polarised Signal*

For a *strictly* monochromatic signal, with cross-spectral matrix $[P]$, it can easily be shown that (Born and Wolf, 1964)

$$\det[P]=0 \tag{2a}$$

$$\mathrm{Tr}[P]=P_{xx}+P_{yy} \tag{2b}$$

$$\sin 2\beta=\frac{2\,\mathrm{Im}(P_{xy})}{P_{xx}+P_{yy}} \tag{2c}$$

$$\tan 2\theta=\frac{2\,\mathrm{Re}(P_{xy})}{P_{xx}-P_{yy}} \tag{2d}$$

$$\gamma_{xy}^2=1 \tag{2e}$$

$$R=1 \tag{2f}$$

(b) *Completely Unpolarised Signal*

Similarly, for a completely unpolarised signal with cross-spectral matrix $[U]$ (Born and Wolf, 1964)

$$U_{xy} = U_{yx} = 0 \tag{3a}$$
$$U_{xx} = U_{yy} \quad (= D \text{ in notation of Fowler et al., 1967}) \tag{3b}$$
$$\gamma_{xy}^2 = 0 \tag{3c}$$
$$\det[U] = D^2 \tag{3d}$$
$$\mathrm{Tr}[U] = 2D \tag{3e}$$
$$R = 0. \tag{3f}$$

(c) Random Noise Signal

For two random noise series, x and y, with cross-spectral matrix $[N]$,

$$N_{xy} = N_{yx} = 0 \quad \text{(Jones, 1977)} \tag{4a}$$
$$\gamma_{xy}^2 = 0 \tag{4b}$$
$$\det[J] = N_{xx} N_{yy} \tag{4c}$$
$$\mathrm{Tr}[J] = N_{xx} + N_{yy} \tag{4d}$$

There is no requirement for $N_{xx} = N_{yy}$ and most data contain unequal noise contributions on each channel. Thus $N_{xx} \neq N_{yy}$ and Born and Wolf (1964) show that the noise exhibits a degree of polarisation (R_r) given by

$$R_r = \frac{N_{xx} - N_{yy}}{N_{xx} + N_{yy}}. \tag{4e}$$

It should be noted that the polarised part of the noise represents a linear polarisation, which is along the x-axis for $R_r > 0$ or along the y-axis for $R_r < 0$.

3. Signal Analysis

Any quasi-monochromatic wave may be regarded as the sum of a completely polarised wave $[P]$ and a completely unpolarised wave $[U]$, which are independent of each other, and this representation is unique

$$J = P + U. \tag{5}$$

Also if several *independent* waves are propagated in the same direction, they superpose and the *total* polarisation matrix $[P_t]$ is given by the summation of the individual matrices $[P_1]$, $[P_2]$, etc.

$$P_t = P_1 + P_2 + \dots \tag{6}$$

and similarly for $[U_t]$.

(a) Uncontaminated Signal

If the data are uncontaminated by noise, then they consist of only two parts, a completely polarised signal [P] and a completely unpolarised signal [U] such that,

$$J=\begin{bmatrix}P_{xx} & P_{xy}\\ P_{yx} & P_{yy}\end{bmatrix}+\begin{bmatrix}D & 0\\ 0 & D\end{bmatrix} \tag{7}$$

and $D=\frac{1}{2}\mathrm{Tr}\,[J]-\frac{1}{2}(\mathrm{Tr}^2\,[J]-4\det[J])^{1/2}$

i.e., the characteristic root (or eigenvalue) of the coherence matrix. The polarisation parameters R, θ, β are given by

$$R \quad =\left(1-\frac{4\det[J]}{\mathrm{Tr}^2\,[J]}\right)^{1/2} \tag{8a}$$

$$\tan 2\theta=\frac{2\,\mathrm{Re}\,(J_{xy})}{J_{xx}-J_{yy}} \tag{8b}$$

$$\sin 2\beta=\frac{2\,\mathrm{Im}\,(J_{xy})}{[(J_{xx}-J_{yy})^2+4J_{xy}\,J_{yx}]^{1/2}}. \tag{8c}$$

These three polarisation parameters – R, θ and β – it should be noted, are *rotational invariants* (see below).

(b) Signal Contaminated by Noise

When there are noise components in the data, N_{xx} and N_{yy}, then the data are represented by a separation into completely polarised signal [P], completely unpolarised signal [U] and noise [N] such that

$$J=\begin{bmatrix}P_{xx} & P_{xy}\\ P_{yx} & P_{yy}\end{bmatrix}+\begin{bmatrix}D & 0\\ 0 & D\end{bmatrix}+\begin{bmatrix}N_{xx} & 0\\ 0 & N_{yy}\end{bmatrix}. \tag{9}$$

For the unique decomposition of this form however a full a priori knowledge of the noise terms (matrix [N]) is required.

From Sect. 2a–c it is obvious that the *coherent* part of the data is given by [P] whilst the *incoherent* part is given by [U]+[N], i.e.,

$$\text{coherent part} \quad =\begin{bmatrix}P_{xx} & P_{xy}\\ P_{yx} & P_{yy}\end{bmatrix} \tag{10a}$$

$$\text{incoherent part}=\begin{bmatrix}D+N_{xx} & 0\\ 0 & D+N_{yy}\end{bmatrix}. \tag{10b}$$

But from the discussion in 2c it is apparent that, unless the noise contributions on the orthogonal components are equal ($N_{xx}=N_{yy}$), the noise matrix is separable into a polarised part [N_p] and an unpolarised part [N_u]. Thus

$$N = N_p + N_u \tag{11a}$$

$$= \begin{bmatrix} \frac{2R_r N_{xx}}{1+R_r} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1-R_r}{1+R_r} N_{xx} & 0 \\ 0 & N_{yy} \end{bmatrix} \tag{11b}$$

(assuming $R_r > 0$, i.e., $N_{xx} > N_{yy}$)

and the *total* polarised matrix $[P_t]$ is given by

$$P_t = P + N_p \tag{12a}$$

(i.e., total polarised part)

$$= \begin{bmatrix} P_{xx} + \frac{2R_r}{1+R_r} N_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{bmatrix}, \tag{12b}$$

whilst the *total* unpolarised matrix $[U_t]$ is

$$U_t = U + N_u \tag{13a}$$

(i.e., total unpolarised part)

$$= \begin{bmatrix} D + \frac{1-R_r}{1+R_r} N_{xx} & 0 \\ 0 & D + N_{yy} \end{bmatrix}. \tag{13b}$$

From Eqs. (10), (12), and (13) it is seen that the *coherent* part equals the *polarised* part, if *and only if*, $R_r = 0$ (i.e., $N_{xx} = N_{yy}$, the noise contributions on both components are of equal magnitude).

Hence θ, derived from Eq. (8b) will be biassed towards the x-axis for $R_r > 0$, and towards the y-axis for $R_r < 0$. Similarly R, from equation 8a, will be always overestimated for $R_r \neq 0$ (since the geometric mean of any two positive numbers cannot exceed their arithmetic mean) whilst $\tan\beta$ – the ellipticity – will be overestimated for $P_{xx} > P_{yy}$ with $R_r < 0$ and for $P_{xx} < P_{yy}$ with $R_r > 0$, and underestimated for $P_{xx} > P_{yy}$ with $R_r > 0$ and for $P_{xx} < P_{yy}$ with $R_r < 0$.

4. Rotation and Coherence

Rotation of matrix J by angle ϕ results in J' from

$$J' = C J C^t \tag{14}$$

where C is the clockwise Cartesian rotation matrix given by

$$C = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

and C^t is the transpose of C. The polarisation parameters in the new co-ordinate system (R', θ', β') are related to the original parameters by

$$R' = R$$
$$\theta' = \theta - \phi$$

and

$$\beta' = \beta,$$

hence are *rotational invariants*.

However, the coherence function, γ_{xy}^2, between the two components, defined as

$$\gamma_{xy}^2 = \frac{|J_{xy}|^2}{J_{xx} J_{yy}} \tag{15}$$

is a function of rotation. From Eq. (8a) for R, and the definition of γ_{xy}^2 [Eq. (15)], it can easily be shown that the two are inter-related by

$$1 - R^2 = \frac{4 J_{xx} J_{yy}}{(J_{xx} + J_{yy})^2} (1 - \gamma_{xy}^2) \tag{16}$$

and hence

$$R^2 \geqq \gamma_{xy}^2 \tag{17}$$

The equality holds when $4 J_{xx} J_{yy} = (J_{xx} + J_{yy})^2$, which can only be true for $J_{xx} = J_{yy}$.

Expanding Eq. (14) and equating the terms for J'_{xx} and J'_{yy} gives the angle required for maximising γ_{xy}^2 as

$$\tan 2\phi_m = \frac{J_{yy} - J_{xx}}{J_{xy} + J_{yx}}. \tag{18}$$

At angle ϕ_m, the following are true

$$R^2 = \gamma_{xy}^2$$
$$\theta' = \pi/4.$$

Such a rotation does not affect the separation of the coherence matrix into polarised and unpolarised parts (as indicated by the rotational invariance of the polarisation parameters) but *does* affect the separation into coherent and incoherent parts, by maximising the former with respect to the latter (as indicated by γ_{xy}^2 maximum).

Finally, the author would like to add that strong caution is advised when interpreting the estimated value of coherence due to the inherent bias associated with the estimation itself (see Jones, 1977).

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